Obstructions of apex classes of graphs

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joint work with:

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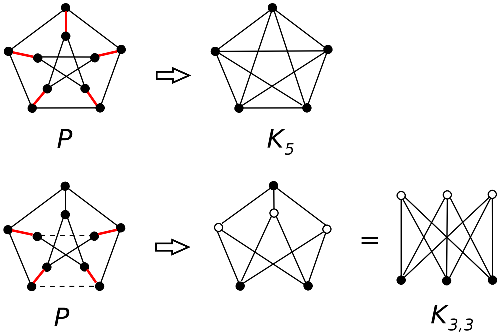
Minors

- If e is an edge of G incident with two distinct vertices u and v, then the contraction of e is the operation of deleting e and identifying u and v.
- Given graphs G and H, we say that H is a minor of G (or that G has an H-minor), denoted by $H \leqslant_{\mathsf{m}} G$ if H can be obtained from G by any sequence of the following operations:
  - deleting an edge;
  - deleting a vertex (and all of its incident edges);
  - contracting an edge.
- Note: The order of operations of deletion and contraction to get a minor of a graph is irrelevant.

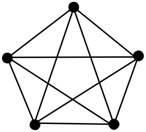
Minors example



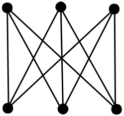
Kuratowski's and Wagner's Theorems

Theorem (Kuratowski '30, Wagner '37)

A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as a minor.



K_5



$K_{3,3}$

Minor-closed classes

- A class $\mathcal{C}$ of graphs is minor-closed if for every $G \in \mathcal{C}$, if $H \leqslant_{\mathsf{m}} G$ then $H \in \mathcal{C}$.

Example

- $\mathcal{P} := \{$planar graphs$\}$
- $\{$projective-planar graphs$\}$
- $\{$toroidal graphs$\}$
- $\{G : G$ is embeddable in $\Sigma\}$, where Σ is a fixed surface
- $\{$linklessly embeddable graphs$\}$
- $\{G : G$ has no H-minor$\}$, where H is a fixed graph
- $\{G : G$ has no K_4-minor$\} = \{$series-parallel graphs$\} = \{G : \mathbf{tw}(G) \leqslant 2\}$
- $\{G : \mathbf{tw}(G) \leqslant k\}$, where k is a fixed positive integer
- $\{G : \mathbf{pw}(G) \leqslant k\}$, where k is a fixed positive integer
- $\mathcal{O} := \{$outerplanar graphs$\}$
- $\mathcal{O}^* := \{$apex-outerplanar graphs$\}$

Excluded minors, Obstruction sets

- Given a graph H and a minor-closed class $\mathcal{C}$, we say that H is an excluded minor of $\mathcal{C}$, if H is a minor-minimal graph not in $\mathcal{C}$ (i.e. $H \notin \mathcal{C}$, but for all $e \in E(H)$, $H \backslash e \in \mathcal{C}$ and $H/e \in \mathcal{C}$).
- For a minor-closed class $\mathcal{C}$, the set of excluded minors of $\mathcal{C}$ is called the obstruction set of $\mathcal{C}$, and is denoted by $\mathbf{ob}(\mathcal{C})$.
- Note: $\mathbf{ob}(\mathcal{C})$ completely characterizes $\mathcal{C}$, because $G \notin \mathcal{C} \iff G$ contains one of the graphs in $\mathbf{ob}(\mathcal{C})$ as a minor (excluded-minor characterization).

Graph Minor Theorem

Theorem (Robertson, Seymour '83 - '10)

(GMT) For any minor-closed class $\mathcal{C}$, $\mathbf{ob}(\mathcal{C})$ is finite.

Proof.

A series of 23 papers totalling several hundred pages published over 27 years... □

Equivalently,

Theorem

In any infinite set of graphs, at least one graph is a minor of another.

Importance of knowing $\mathbf{ob}(\mathcal{C})$

Theorem (Robertson, Seymour '95)

For any fixed graph H, there is an algorithm to determine whether a given n-vertex graph has H as a minor in $O(n^3)$-time.

Consequently,

Corollary

(Membership Testing) For any minor-closed class $\mathcal{C}$, there is an algorithm to determine whether a given n-vertex graph G belongs to $\mathcal{C}$ in $O(n^3)$-time.

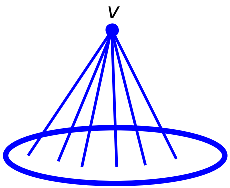
Known excluded-minor characterizations

Example

- **ob**(planar graphs) $= \{K_{3,3}, K_5\}$ (Kuratowski '30, Wagner '37)
- **ob**(series-parallel graphs) $= \{K_4\} =$ **ob**(graphs of tree-width $\leqslant 2$) (Dirac '52)
- |**ob**(projective-planar graphs)| $= 35$ (Archdeacon '81)
- **ob**(graphs of tree-width $\leqslant 3$) $= \{K_5, V_8, Oct, L_5\}$ (Arnborg, Corneil, Proskurowski '90; Satyanarayana, Tung '90)
- |**ob**(linklessly embeddable graphs)| $= 7$ (Robertson, Seymour, Thomas '95)
- **ob**(outerplanar graphs) $= \{K_{2,3}, K_4\}$ (Chartrand, Harary '67)
- |**ob**(α-outerplanar graphs)| $= 13$ (Wargo '96)
- |**ob**(apex-outerplanar graphs)| $= 57$ (Ding, D. '10)

Apex-outerplanar graphs

- A graph is outerplanar if it can be embedded in the plane (with no edges crossing) with all vertices incident to one common face.
- A graph G is apex-outerplanar if there exists $v \in V(G)$ such that $G - v$ is outerplanar. Such a vertex v is called an apex vertex of G.

$G =$ 

- We let $\mathcal{O}$ and $\mathcal{O}^*$ denote the classes of outerplanar and apex-outerplanar graphs, respectively.
- Note: Since having loops or parallel edges has no impact on (apex-) outerplanarity, all graphs in the remainder of the talk are assumed to be simple.

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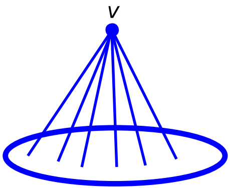
$G =$ 

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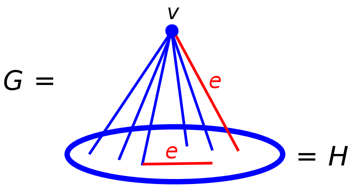
General Framework

- Let $\mathcal{C}$ be a minor-closed class of graphs, and let $\mathcal{C}^*$ be the class of graphs that contain a vertex whose removal leaves a graph in $\mathcal{C}$.
- Note: $\mathcal{C} \subseteq \mathcal{C}^*$, and $\mathcal{C}^*$ is minor-closed



- General Problem:
  Given: a minor-closed class $\mathcal{C}$ and $\textbf{ob}(\mathcal{C})$
  Find: $\textbf{ob}(\mathcal{C}^*)$
- Adler, Grohe, Kreutzer ('08) showed that this problem is computable...
- Already very hard for $\mathcal{C} = \mathcal{P} = \{\text{planar graphs}\}$.

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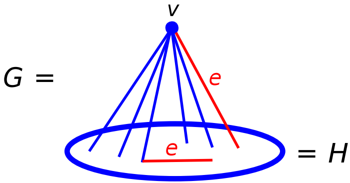

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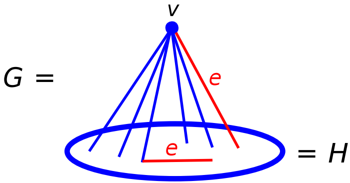

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Motivation: Apex-planar graphs

Theorem (Robertson, Seymour '03)

(Structure Theorem) If $\mathcal{C}$ is a proper minor-closed class of graphs, then every graph in $\mathcal{C}$ is glued together in a tree-like fashion from graphs that can be nearly embedded in a fixed surface.

Conjecture (Hadwiger '43)

Graphs with no K_n-minor can be colored with at most $(n-1)$ colors.

- The case $n = 5$ is equivalent to the Four Color Theorem.
- The case $n = 6$ was proved by Robertson, Seymour, and Thomas '93.
- RST proved that every minimal counterexample to Hadwiger's for $n = 6$ is apex-planar, so no counterexample exists (by 4CT).

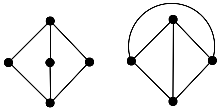
Conjecture (Jorgensen '94)

Every 6-connected graph with no K_6-minor is apex planar.

- Implies Hadwiger's for $n = 6$.
- Proved for large graphs by DeVos, Hegde, Kawarabayashi, Norin, Thomas, Wollan.

Obstructions of apex-outerplanar graphs

- General Problem:
  Given: a minor-closed class $\mathcal{C}$ and $\mathbf{ob}(\mathcal{C})$
  Find: $\mathbf{ob}(\mathcal{C}^*)$
- Already very hard for $\mathcal{C} = \mathcal{P} = \{$planar graphs$\}$.
- So what about $\mathcal{C} = \mathcal{O} = \{$outerplanar graphs$\}$?
- $\mathbf{ob}(\mathcal{O}) = \{K_{2,3}, K_4\}$ (Chartrand, Harary '67)



- Problem: Find: $\mathbf{ob}(\mathcal{O}^*)$.

Summary

$\mathcal{C} = \{$cactus graphs$\}$

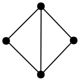
$\mathcal{O} = \{$outerplanar graphs$\}$

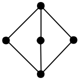
$\mathcal{S} = \{$series-parallel graphs$\}$

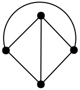
$\textbf{ob}(\mathcal{C}) = \{K_4 - e\}$

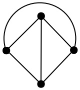
$\textbf{ob}(\mathcal{O}) = \{K_{2,3}, K_4\}$

$\textbf{ob}(\mathcal{S}) = \{K_4\}$









$\mathcal{C}^* = \{$apex-cactus graphs$\}$

$\mathcal{O}^* = \{$apex-outerplanar graphs$\}$

$\mathcal{S}^* = \{$apex-series-parallel graphs$\}$

$|\textbf{ob}(\mathcal{C}^*)| = 25$

(D. '13)

$|\textbf{ob}(\mathcal{O}^*)| = 57$

(Ding, D. '10)

$|\textbf{ob}(\mathcal{S}^*)| \geq 16$

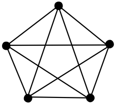
(in progress...)

The Starting Lineup

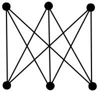
- **GOAL:** To find $\mathbf{ob}(\mathcal{O}^*)$

The Starting Lineup

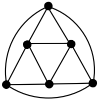
- **GOAL:** To find $\mathbf{ob}(\mathcal{O}^*)$
- The following graphs all belong to $\mathbf{ob}(\mathcal{O}^*)$.



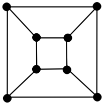
K_5



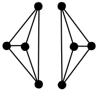
$K_{3,3}$



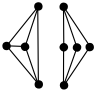
Oct



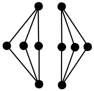
Q



$2K_4$



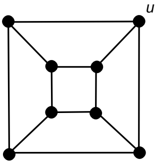
$K_4 \mid K_{2,3}$



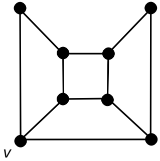
$2K_{2,3}$

- For each G above, we need to check:
  $G \notin \mathcal{O}^*$ (i.e. G has an apex vertex)
  $\forall e \in E(G), G\backslash e \in \mathcal{O}^*$ and $G/e \in \mathcal{O}^*$ (i.e. $G\backslash e$ and G/e don't have an apex vertex)

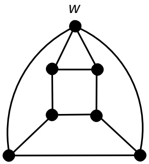
Minor-minimality of Q



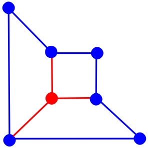
Q



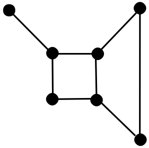
$Q \setminus e$



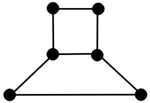
Q / e



$Q - u$

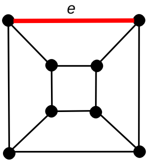


$(Q \setminus e) - v$

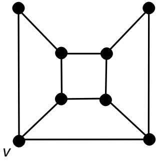


$(Q / e) - w$

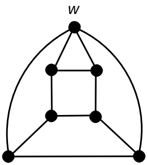
Minor-minimality of Q



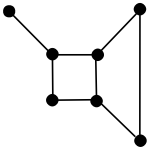
Q



$Q \setminus e$



Q / e



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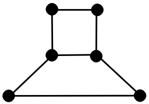


$(Q / e) - w$

Connectivity is 2 or 3

Let $G \in \mathbf{ob}(\mathcal{O}^*) - \{K_5, K_{3,3}, Oct, Q, 2K_4, K_4|K_{2,3}, 2K_{2,3}\}$. Then

(1) G is planar

(2) G is 2-connected

(3) G is not 4-connected

Proof.

(1) G has no $\{K_5, K_{3,3}\}$-minor.

(2) Follows from the fact that G does not contain any of the graphs: $2K_4$, $K_4|K_{2,3}$, $2K_{2,3}$ as a minor.

(3) G is 4-connected $\Longrightarrow \delta(G) \geqslant 4 \Longrightarrow G \geqslant_{\mathrm{m}} K_5$ or $G \geqslant_{\mathrm{m}} Oct$ (Halin, Jung '63), a contradiction.

□

- $\kappa(G) = 2$ or 3

Connectivity-Three Case

We prove:

Lemma

If G is 3-connected in $\textbf{ob}(\mathcal{O}^*)$, then $G \in \{K_5, K_{3,3}, Oct, Q\}$.

We use the following fact:

Lemma

If G is 3-connected and $|G| > 4$, then G has an edge e such that G/e is also 3-connected.

- Such an edge is called contractible.

Connectivity-Three Case

Denote by v_{xy} the vertex formed by contracting edge xy. Since G is minor-minimal $\notin \mathcal{O}^*$, there are two possibilities:

- Case 1: There exists a contractible edge $xy \in E(G)$ such that v_{xy} is not apex in G/xy (and hence, there exists an apex vertex $a \neq v_{xy}$ in G/xy).
- Case 2: For every contractible edge $xy \in E(G)$, v_{xy} is an apex vertex in G/xy.

By assuming that $G \in \textbf{ob}(\mathcal{O}^*) - \{K_5, K_{3,3}, Oct, Q, 2K_4, K_4|K_{2,3}, 2K_{2,3}\}$, we show that we reach a contradiction in each case, proving the Lemma.

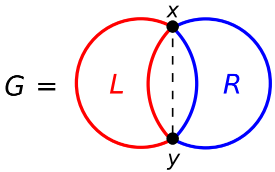
Connectivity-Two Case

Let G be a graph and $x, y \in V(G)$. A 2-separation of G over $\{x, y\}$ is a pair of induced subgraphs (L, R) of G such that:

(1) $E(L) \cup E(R) = E(G)$;

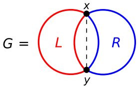
(2) $V(L) \cup V(R) = V(G)$ and $V(L) \cap V(R) = \{x, y\}$;

(3) $V(L) - V(R) \neq \varnothing$ and $V(R) - V(L) \neq \varnothing$.



- Note: $\{x, y\}$ is a 2-vertex-cut of G.

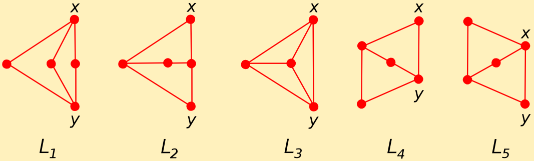
Connectivity-Two Case: Key Lemma



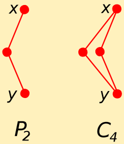
Lemma

Let (L, R) *be a* 2*-separation of* G *over vertices* $\{x, y\}$*.*

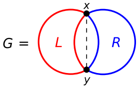
(1) *If* $L \notin \mathcal{O}$ *and* $R \notin \mathcal{O}$*, then one of* L *or* R *is one of the five graphs:* L_1*,* L_2*,* L_3*,* L_4*, or* L_5*.*



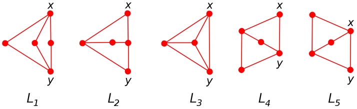
(2) *If* $L \in \mathcal{O}$*, then* $xy \notin E(G)$ *and* $L = P_2$ *or* C_4*.*



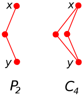
Connectivity-Two Case: Roadmap



- Case 1: There exists a 2-separation such that both $L \notin \mathcal{O}$ and $R \notin \mathcal{O}$



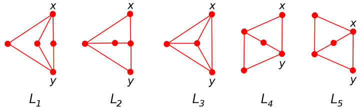
- Case 2: For each 2-separation, $L = P_2$ or C_4



  - Case 2.1: There exists a 2-separation such that $L = C_4$
    - Case 2.1.1: There exists a 2-separation such that $L = C_4$ and $R - \{x, y\} \notin \mathcal{O}$
    - Case 2.1.2: There exists a 2-separation such that $L = C_4$ and for every such 2-separation $R - \{x, y\} \in \mathcal{O}$
  - Case 2.2: For each 2-separation, $L = P_2$
    - Case 2.2.1: There exists a 2-separation such that $L = P_2$ and $R - \{x, y\} \notin \mathcal{O}$
    - Case 2.2.2: For each 2-separation, $L = P_2$ and $R - \{x, y\} \in \mathcal{O}$

Case 1

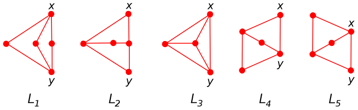
- Case 1: There exists a 2-separation such that both $L \notin \mathcal{O}$ and $R \notin \mathcal{O}$



We yield the T-family $\subseteq$ **ob**$(\mathcal{O}^*)$:

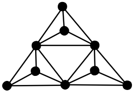
Case 1

- Case 1: There exists a 2-separation such that both $L \notin \mathcal{O}$ and $R \notin \mathcal{O}$

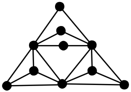


We yield the T-family $\subseteq$ **ob**$(\mathcal{O}^*)$:

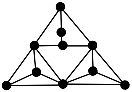
Case $1 \implies$ T-family



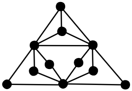
T_1



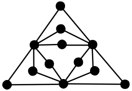
T_2



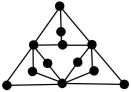
T_3



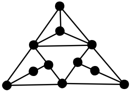
T_4



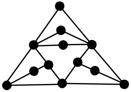
T_5



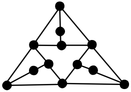
T_6



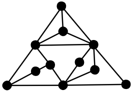
T_7



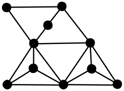
T_8



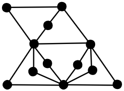
T_9



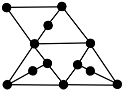
T_{10}



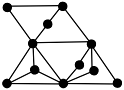
T_{11}



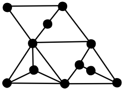
T_{12}



T_{13}

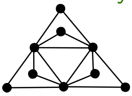


T_{14}

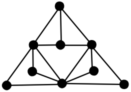


T_{15}

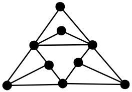
Case 1 $\implies$ T-family



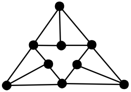
T_5



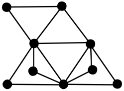
T_6



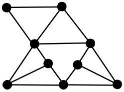
T_8



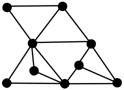
T_9



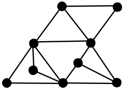
T_{12}



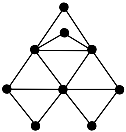
T_{13}



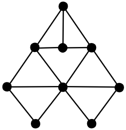
T_{16}



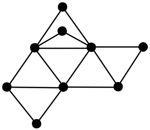
T_{19}



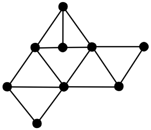
T_{21}



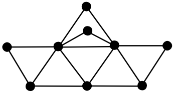
T_{22}

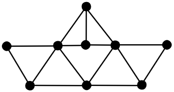


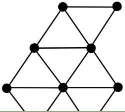
T_{24}

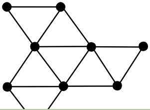


T_{25}



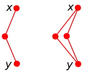






Case 2.1.1 $\Longrightarrow$ G-family

- Case 2: For each 2-separation, $L = P_2$ or C_4

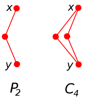


  - Case 2.1: There exists a 2-separation such that $L = C_4$
    - Case 2.1.1: There exists a 2-separation such that $L = C_4$ and $R - \{x, y\} \notin \mathcal{O}$

We yield the G-family $\subseteq$ **ob**$(\mathcal{O}^*)$:

Case $2.1.1 \implies$ G-family

- Case 2: For each 2-separation, $L = P_2$ or C_4

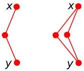


  - Case 2.1: There exists a 2-separation such that $L = C_4$
    - Case 2.1.1: There exists a 2-separation such that $L = C_4$ and $R - \{x, y\} \notin \mathcal{O}$

We yield the G-family $\subseteq$ **ob**$(\mathcal{O}^*)$:

Case $2.1.1 \implies$ G-family

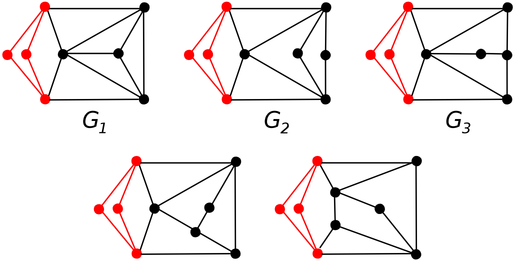
- Case 2: For each 2-separation, $L = P_2$ or C_4



P_2 C_4

  - Case 2.1: There exists a 2-separation such that $L = C_4$
    - Case 2.1.1: There exists a 2-separation such that $L = C_4$ and $R - \{x, y\} \notin \mathcal{O}$

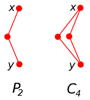
We yield the G-family $\subseteq$ **ob**$(\mathcal{O}^*)$:



G_1 G_2 G_3 G_4 G_5

Case $2.1.2 \implies$ J-family

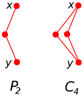
- Case 2: For each 2-separation, $L = P_2$ or C_4



  - Case 2.1: There exists a 2-separation such that $L = C_4$
    - Case 2.1.2: There exists a 2-separation such that $L = C_4$ and for every such 2-separation $R - \{x, y\} \in \mathcal{O}$

Case 2.1.2 $\Longrightarrow$ J-family

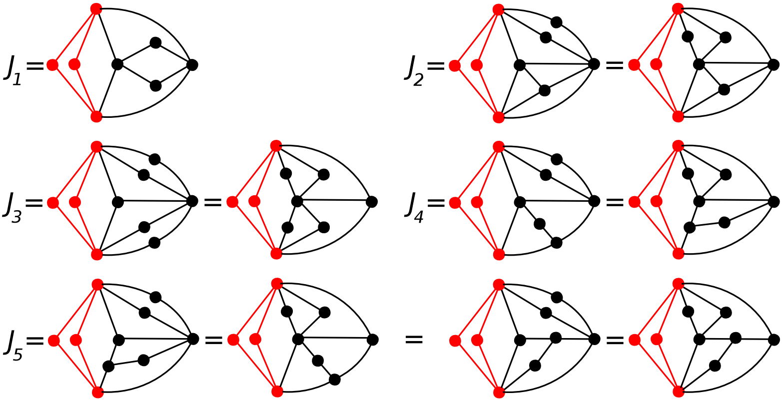
- Case 2: For each 2-separation, $L = P_2$ or C_4



  - Case 2.1: There exists a 2-separation such that $L = C_4$
    - Case 2.1.2: There exists a 2-separation such that $L = C_4$ and for every such 2-separation $R - \{x, y\} \in \mathcal{O}$

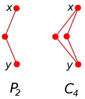
We yield the J-family $\subseteq$ **ob**$(\mathcal{O}^*)$:

Case $2.1.2 \implies$ J-family



Case $2.2.1 \implies$ H-family

- Case 2: For each 2-separation, $L = P_2$ or C_4

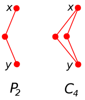


  - Case 2.2: For each 2-separation, $L = P_2$
    - Case 2.2.1: There exists a 2-separation such that $L = P_2$ and $R - \{x, y\} \notin \mathcal{O}$

We yield the H-family $\subseteq$ **ob**$(\mathcal{O}^*)$:

Case $2.2.1 \implies$ H-family

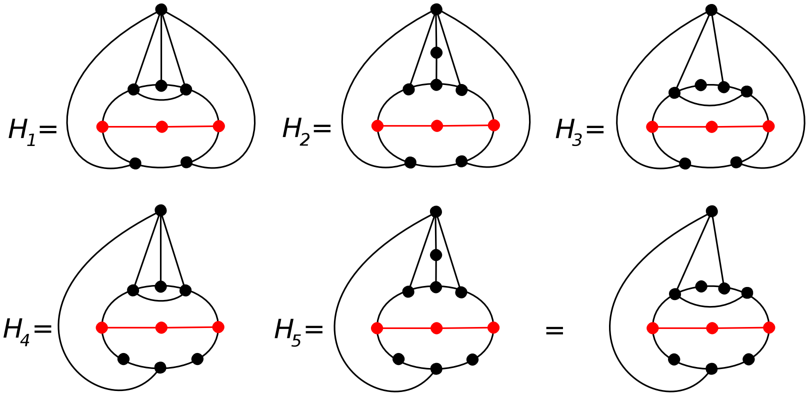
- Case 2: For each 2-separation, $L = P_2$ or C_4



  - Case 2.2: For each 2-separation, $L = P_2$
    - Case 2.2.1: There exists a 2-separation such that $L = P_2$ and $R - \{x, y\} \notin \mathcal{O}$

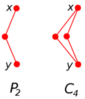
We yield the H-family $\subseteq$ **ob**$(\mathcal{O}^*)$:

Case $2.2.1 \Longrightarrow$ H-family



Case $2.2.2 \implies$ Q-family

- Case 2: For each 2-separation, $L = P_2$ or C_4

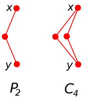


  - Case 2.2: For each 2-separation, $L = P_2$
    - Case 2.2.2: For each 2-separation, $L = P_2$ and $R - \{x, y\} \in \mathcal{O}$

We yield the Q-family $\subseteq$ **ob**$(\mathcal{O}^*)$:

Case $2.2.2 \implies$ Q-family

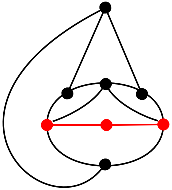
- Case 2: For each 2-separation, $L = P_2$ or C_4

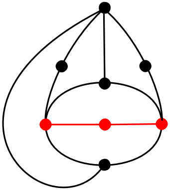


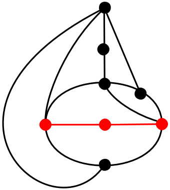
  - Case 2.2: For each 2-separation, $L = P_2$
    - Case 2.2.2: For each 2-separation, $L = P_2$ and $R - \{x, y\} \in \mathcal{O}$

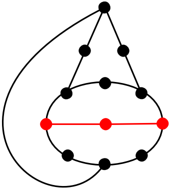
We yield the Q-family $\subseteq$ **ob**$(\mathcal{O}^*)$:

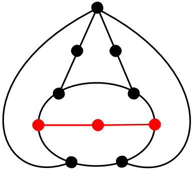
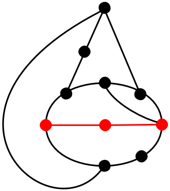
Case $2.2.2 \implies$ Q-family

$Q_1 =$ 

$Q_2 =$ 

$Q_3 =$ 

$Q_4 =$ 

$Q_5 =$  $=$ 

Main Theorem

Theorem (Ding, D. '10)

$\mathbf{ob}(\mathcal{O}^*) = \{K_5, K_{3,3}, Oct, Q, 2K_4, K_4|K_{2,3}, 2K_{2,3}\} \cup \mathcal{T} \cup \mathcal{G} \cup \mathcal{J} \cup \mathcal{H} \cup \mathcal{Q}$

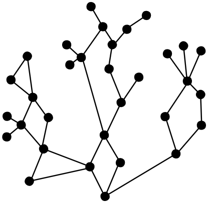
where:

- $\mathcal{T} := \{T_1, \ldots, T_{30}\}$ (T-family)
- $\mathcal{G} := \{G_1, G_2, G_3, G_4, G_5\}$ (G-family)
- $\mathcal{J} := \{J_1, J_2, J_3, J_4, J_5\}$ (J-family)
- $\mathcal{H} := \{H_1, H_2, H_3, H_4, H_5\}$ (H-family)
- $\mathcal{Q} := \{Q_1, Q_2, Q_3, Q_4, Q_5\}$ (Q-family)

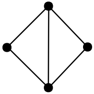
Obstructions of apex-cactus graphs

- General Problem:
  Given: a minor-closed class $\mathcal{C}$ and $\mathbf{ob}(\mathcal{C})$
  Find: $\mathbf{ob}(\mathcal{C}^*)$

- What about $\mathcal{C} = \{\text{cactus graphs}\}$?



- $\mathbf{ob}(\mathcal{C}) = \{D := K_4 - e\}$ (El-Mallah, Colbourn '88)

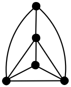


- Problem: Find: $\mathbf{ob}(\mathcal{C}^*)$.

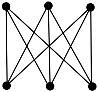
The Starting Lineup

- **GOAL:** To find **ob**$(\mathcal{C}^*)$

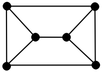
- **GOAL:** To find **ob**$(\mathcal{C}^*)$
- The following graphs all belong to **ob**$(\mathcal{C}^*)$.



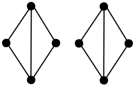
K_5 - e



$K_{3,3}$



Prism



2D

- For each G above, we need to check:
  $G \notin \mathcal{C}^*$ (i.e. G has an apex vertex)
  $\forall e \in E(G), G \backslash e \in \mathcal{C}^*$ and $G/e \in \mathcal{C}^*$ (i.e. $G \backslash e$ and G/e don't have an apex vertex)

Connectivity is 2 or 3

Let $G \in \mathbf{ob}(\mathcal{C}^*) - \{K_5 - e, K_{3,3}, Prism, 2D\}$. Then

(1) G is planar

(2) G is 2-connected

(3) G is not 4-connected

Proof.

(1) G has no $\{K_5, K_{3,3}\}$-minor, because it has no $\{K_5 - e, K_{3,3}\}$-minor

(2) Follows from the fact that G does not contain $2D$ as a minor.

(3) G is 4-connected $\Longrightarrow \delta(G) \geqslant 4 \Longrightarrow G \geqslant_{\mathrm{m}} K_5$ or $G \geqslant_{\mathrm{m}} Oct$ (Halin, Jung '63), hence $G \geqslant_{\mathrm{m}} K_5 - e$, a contradiction.

□

- $\kappa(G) = 2$ or 3

Connectivity-Three Case

We prove:

Lemma

If G is 3-connected in **ob**$(\mathcal{C}^*)$, *then* $G \in \{K_5 - e, K_{3,3}, Prism\}$.

We use the following fact:

Lemma

If G is 3-connected and $|G| > 4$, then G has an edge e such that G/e is also 3-connected.

- Such an edge is called contractible.

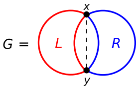
Connectivity-Three Case

Denote by v_{xy} the vertex formed by contracting edge xy. Since G is minor-minimal $\notin \mathcal{C}^*$, there are two possibilities:

- Case 1: There exists a contractible edge $xy \in E(G)$ such that v_{xy} is not apex in G/xy (and hence, there exists an apex vertex $a \neq v_{xy}$ in G/xy).
- Case 2: For every contractible edge $xy \in E(G)$, v_{xy} is an apex vertex in G/xy.

By assuming that $G \in \mathbf{ob}(\mathcal{O}^*) - \{K_5 - e, K_{3,3}, Prism, 2D\}$, we show that we reach a contradiction in each case, proving the Lemma.

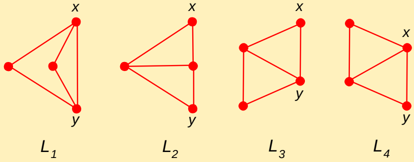
Connectivity-Two Case: New Key Lemma



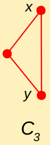
Lemma

Let (L, R) *be a* 2*-separation of* G *over vertices* $\{x, y\}$.

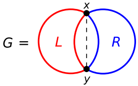
(1) *If* $L \notin \mathcal{C}$ *and* $R \notin \mathcal{C}$, *then one of* L *or* R *is one of the four graphs:* L_1, L_2, L_3, *or* L_4.



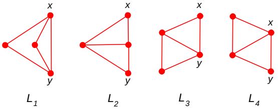
(2) *If* $L \in \mathcal{C}$, *then* $L = C_3$.



Connectivity-Two Case: Roadmap



- Case 1: There exists a 2-separation such that both $L \notin \mathcal{C}$ and $R \notin \mathcal{C}$

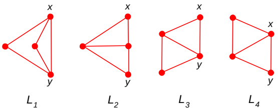


- Case 2: For each 2-separation, $L = C_3$

  - Case 2.1: There exists a 2-separation such that $L = C_3$ and $R - \{x, y\} \notin \mathcal{C}$
  - Case 2.2: For each 2-separation, $L = C_3$ and $R - \{x, y\} \in \mathcal{C}$

Case 1

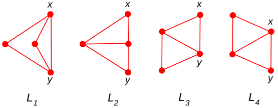
- Case 1: There exists a 2-separation such that both $L \notin \mathcal{C}$ and $R \notin \mathcal{C}$



We yield the T-family $\subseteq$ **ob**$(\mathcal{C}^*)$:

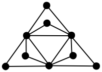
Case 1

- Case 1: There exists a 2-separation such that both $L \notin \mathcal{C}$ and $R \notin \mathcal{C}$

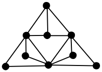


We yield the T-family $\subseteq$ **ob**$(\mathcal{C}^*)$:

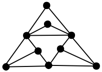
Case $1 \implies$ T-family



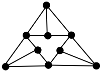
T_5



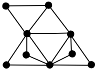
T_6



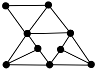
T_8



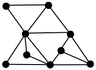
T_9



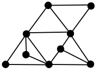
T_{12}



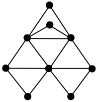
T_{13}



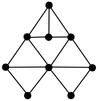
T_{16}



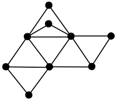
T_{19}



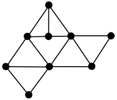
T_{21}



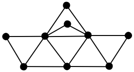
T_{22}



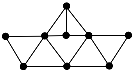
T_{24}



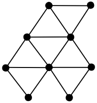
T_{25}



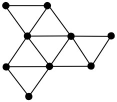
T_{27}



T_{28}



T_{29}



T_{30}

Case 2.1 No $\implies$ G-family

- Case 2: For each 2-separation, $L = C_3$

  - Case 2.1: There exists a 2-separation such that $L = C_3$ and $R - \{x, y\} \notin \mathcal{C}$

This case does not yield any new graphs in **ob**$(\mathcal{C}^*)$:

Case 2.1 No $\implies$ G-family

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C_3

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This case does not yield any new graphs in **ob**$(\mathcal{C}^*)$:

Case $2.2 \Longrightarrow$ J-family

- Case 2: For each 2-separation, $L = P_2$ or C_4

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Case $2.2 \implies$ J-family

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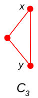

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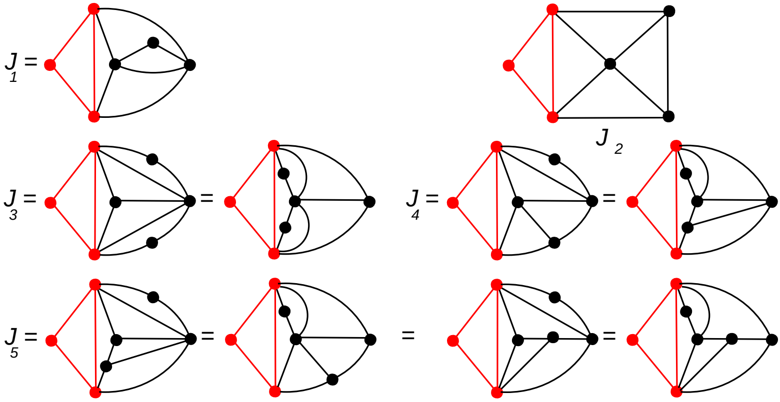
We yield the J-family $\subseteq$ **ob**$(\mathcal{C}^*)$:

Case $2.2 \Longrightarrow$ J-family



Main Theorem

Theorem (D. '13)

$\mathbf{ob}(\mathcal{C}^*) = \{K_5 - e, K_{3,3}, Prism, 2D\} \cup \mathcal{T} \cup \mathcal{J}$

where:

- $\mathcal{T} := \{T_1, \ldots, T_{16}\}$ (T-family)
- $\mathcal{J} := \{J_1, J_2, J_3, J_4, J_5\}$ (J-family)

Apex-series-parallel graphs

- General Problem: Given: a minor-closed class $\mathcal{C}$ and $\mathbf{ob}(\mathcal{C})$, find: $\mathbf{ob}(\mathcal{C}^*)$.
- Already very hard for $\mathcal{C} = \mathcal{P} = \{\text{planar graphs}\}$, but what about $\mathcal{C} = \mathcal{S} = \{\text{series-parallel graphs}\}$?
- $\mathbf{ob}(\mathcal{S}^*)$ includes the following graphs:

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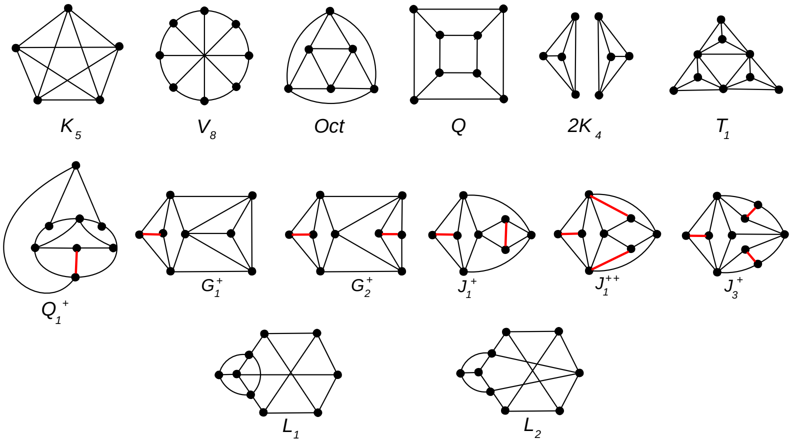


The hunt for $\mathbf{ob}(\mathcal{S}^*)$

Observation 1

Let $G \in \mathbf{ob}(\mathcal{S}^*)$. Then $G \notin \mathcal{S}^*$, thus $G \notin \mathcal{O}^*$, and so $G \geqslant_m H$ for some $H \in \mathbf{ob}(\mathcal{O}^*)$.

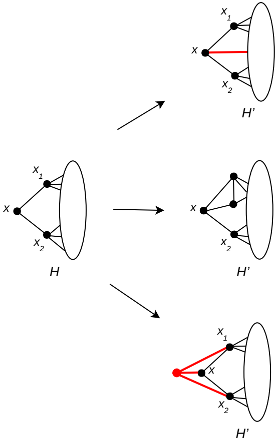
Observation 2

If $G \in \mathbf{ob}(\mathcal{S}^*)$, then $\delta(G) \geqslant 3$.

Helpful Lemma

Lemma

Let H be a 2-connected minor of a 2-connected graph G with $\delta(G) \geqslant 3$. Let $x \in V(H)$ with $deg_H(x) = 2$, and let x_1 and x_2 be its two neighbors with $deg_H(x_i) \geqslant 3$ for $i = 1, 2$. Then $G \succeq H'$, where H' is obtained from H by one of the following operations:



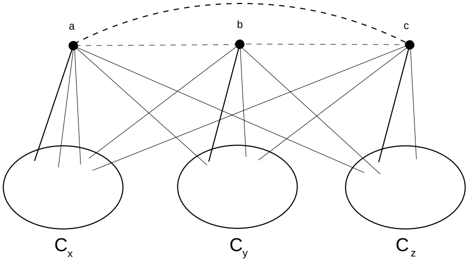
Helpful Theorem

Theorem (Kezdy, McGuiness '91)

Suppose that G is a 3-connected graph containing a $K_{3,3}$-subdivision with branch vertices $\{\{a, b, c\}, \{x, y, z\}\}$. Then at least one of the following must hold:

- $G \geqslant_m K_5$
- $\{\{a, b, c\}\}$ *separates* G *such that* x, y, *and* z *are in separate components.*
- $\{\{x, y, z\}\}$ *separates* G *such that* a, b, *and* c *are in separate components.*
- $G = V_8$

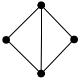
So suppose $G \in \mathbf{ob}(\mathcal{S}^*)$ is 3-connected and contains a $K_{3,3}$-subdivision:



Summary

$\mathcal{C} = \{$cactus graphs$\}$

$\textbf{ob}(\mathcal{C}) = \{K_4 - e\}$



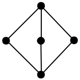
$\mathcal{C}^* = \{$apex-cactus graphs$\}$

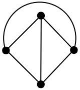
$|\textbf{ob}(\mathcal{C}^*)| = 25$

(D. '13)

$\mathcal{O} = \{$outerplanar graphs$\}$

$\textbf{ob}(\mathcal{O}) = \{K_{2,3}, K_4\}$





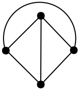
$\mathcal{O}^* = \{$apex-outerplanar graphs$\}$

$|\textbf{ob}(\mathcal{O}^*)| = 57$

(Ding, D. '10)

$\mathcal{S} = \{$series-parallel graphs$\}$

$\textbf{ob}(\mathcal{S}) = \{K_4\}$



$\mathcal{S}^* = \{$apex-series-parallel graphs$\}$

$|\textbf{ob}(\mathcal{S}^*)| \geq 16$

(in progress...)

Open Questions

Questions

- *For what other classes $\mathcal{C}$ can the problem of finding $\mathbf{ob}(\mathcal{C}^*)$ be solved?*
- *Given $\mathcal{C}$ and $\mathbf{ob}(\mathcal{C})$, can we give an upper bound on the sizes of the graphs in $\mathbf{ob}(\mathcal{C}^*)$?*

Thank You!