Discrete Fair Division Using Posets

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Outline:

- From a linear order on items to a partial order on sets of items
- Partial orders as ballots
- Some questions
- References

Fair Division:

- k items to be distributed to n players.
- Items are desirable (goods, not bads).
- Items are indivisible (e.g., houses or cars, not money or land).
- Each player has linear preferences on the items.
- Players may (and hopefully will) rank items differently.

An *allocation* of items $[k] \equiv \{1, \ldots, k\}$ to players $\{1, \ldots, n\}$ is a function $\alpha : [k] \to [n]$.

Goal: use each player's linear preferences to impose an order on the set of allocations.

Assumption: to rank an allocation α relative to β, player p compares $\alpha^{-1}(\{p\})$ with $\beta^{-1}(\{p\})$ and is indifferent to other players' spoils.

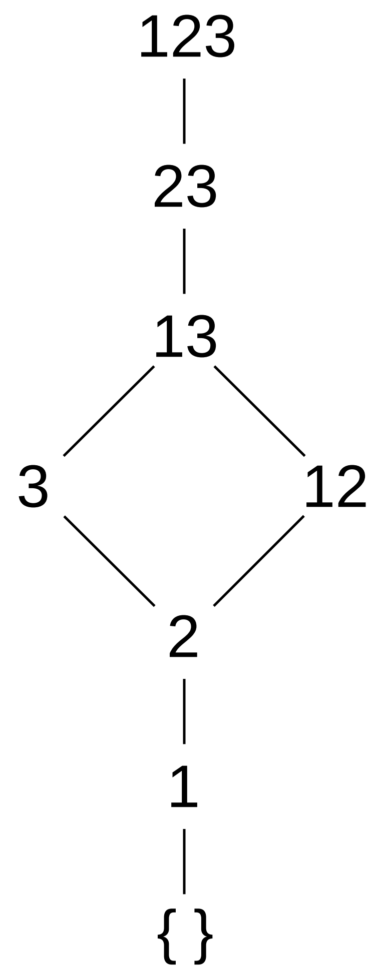
Thus, it suffices to impose an order on $\mathcal{P}([k])$, the collection of sets of goods.

One approach: $S \leq T$ if T can be obtained from S by trading some elements of S for items that are at least as good, then adding some additional items.

That is, $S \leq T$ if there exists an injection $\phi : S \to T$ such that for each $s \in S$, the player ranks $\phi(s)$ at least as high as s.

Suppose a particular player holds preferences $1 < \cdots < k$ on $[k]$. Then, for example, $\{1, 4, 5\} < \{2, 3, 5, 6\}$, but $\{1, 4, 5\}$ and $\{2, 3, 5\}$ are incomparable.

Call the resulting poset $f_p(k)$.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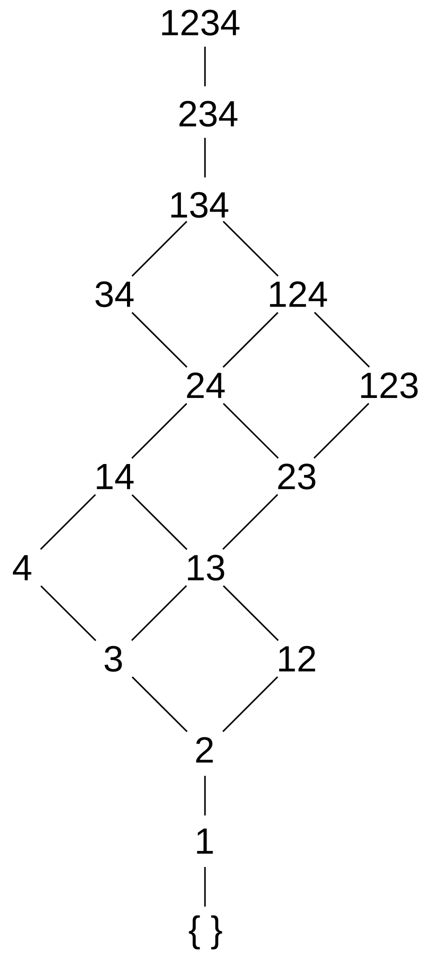
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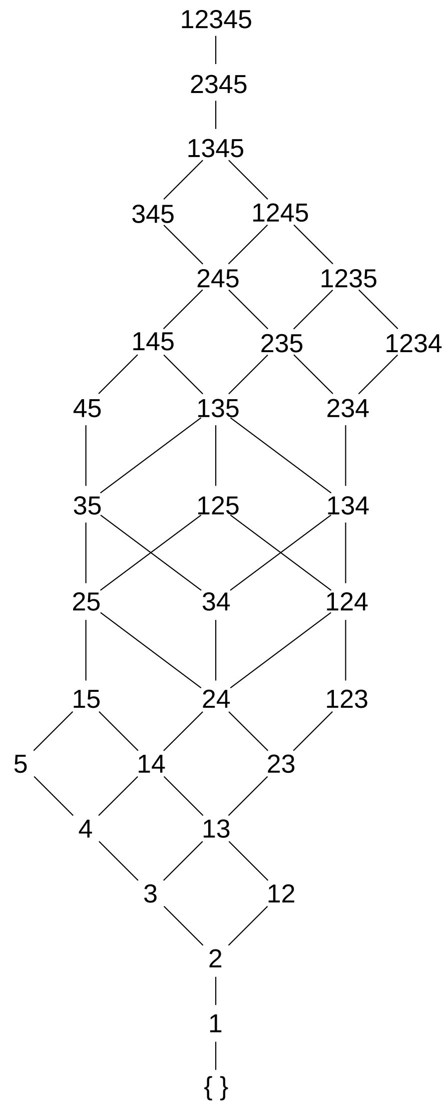
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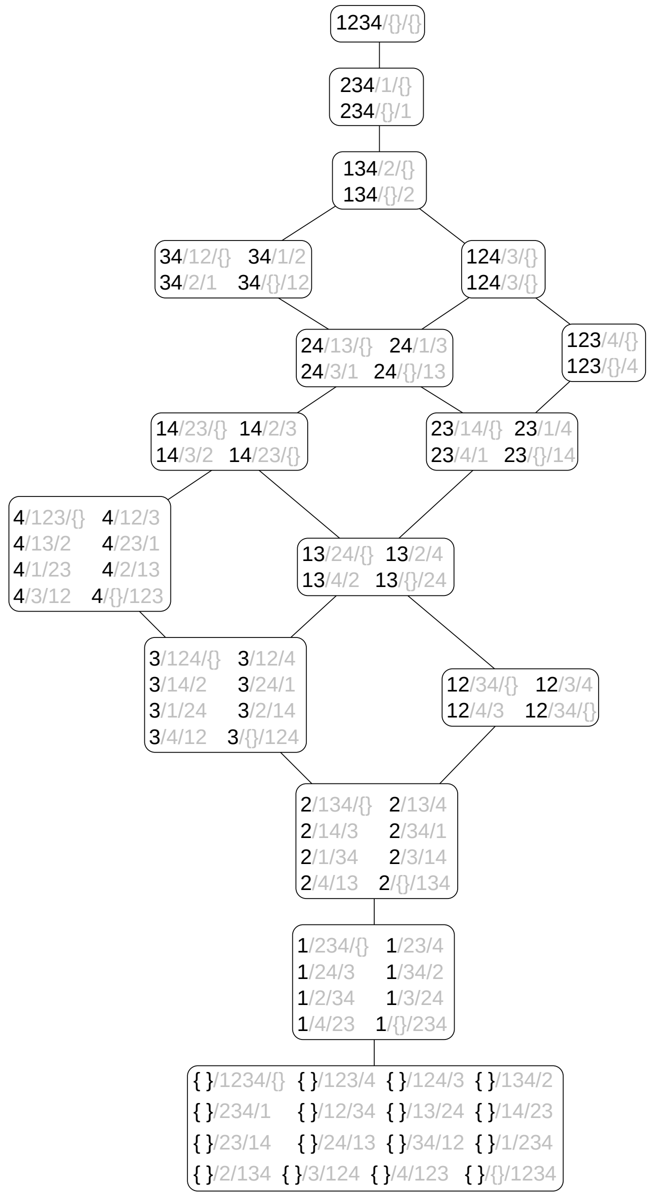
$f_p(k)$ is the Bruhat order on the quotient of a free Coxeter group by a maximal parabolic subgroup. It has been studied by Stanley (1980), Lindström (1970), and others.

Hopkins & Jones (2009): made reference to a subposet of this one in the context of a fair division procedure for two players.

$f_p(k)$ can be extended to a partial order $F_p(k)$ on the set of allocations.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Idea: use $F_p(k)$ as p's ballot in an election where the candidates are allocations.

Voting methods: Plurality, Approval, Borda, many others. Most assume linear orders or (some kinds of) linear orders with ties.

How to vote with partially ordered ballots?

Ackerman, Choi, Coughlin, G., Wood (2013) offered a couple of approaches. Linear extensions, statistical sampling.

Cullinan, Hsiao, Polett (2013) offered a form of Borda count on posets:

$p(a) = 2 * L(a) + I(a)$, where $L(a)$ is the number of elements less than a and $I(a)$ is the number of elements incomparable to a.

Properties:

- The unique (up to affine transformation) constant total weight procedure that is linear in L and I.
- The unique social choice function that is consistent, faithful, neutral, and has the cancellation property.
- Monotone.
- Pareto.
- Fails to satisfy plurality.

Questions:

- Does the CHP approach give a different order (other than ties) from the usual weights approach?
- If we assume that the players' linear orders on items are far enough apart on the permutahedron, can we guarantee good CHP outcomes?

References:

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[4] Lindström, B. "Conjecture on a theorem similar to Sperner's." *Combinatorial Structures and their Applications*, Guy, Hanani,

Sauer, and Schonheim, eds. Gordon and Breach, New York, 1970, 241.

[5] Stanley, R. “Weyl groups, the hard Lefschetz theorem, and the Sperner property.” SIAM J. Alg. Disc. Meth., Vol. 1, No. 2 (1980): 168 - 184.