Coloring Graphs Drawn with No Dependent Crossings

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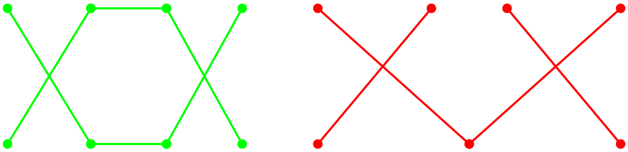
Crossings

- Given a drawing of a graph, a pair of crossed edges is called a *crossing*.
- Two crossings are *independent* if their vertex sets are pairwise disjoint.
- Two crossings that are not independent are called *dependent*.



Result

Theorem (Král and Stacho)

If G is a graph drawn in the plane with no dependent crossings, then G is 5-colorable.

What about other surfaces?

Heawood's Formula

Theorem (Heawood)

Let G be a graph embedded in a nonplanar surface Σ. Then $\chi(G) \leq h(\Sigma) = \lfloor \frac{7+\sqrt{49-24\varepsilon(\Sigma)}}{2} \rfloor$.

How do crossings affect this result?

Main Results

Theorem

Suppose G is a graph drawn on a nonplanar surface Σ with no dependent crossings. Then $\chi(G) \leq h+1$.

Can we do better?

Where Does K_{h+1} Fit?

Theorem

If G is a graph drawn on a nonplanar surface Σ with no dependent crossings, then $\chi(G) \leq h$.

Where Does K_{h+1} Fit?

Conjecture

If G is a graph drawn on a nonplanar surface Σ with no dependent crossings, then $\chi(G) \leq h$.

We can show that K_8, K_9 and K_{10} cannot be drawn with no dependent crossings on surfaces with Heawood numbers of 7, 8 and 9, respectively.
Moreover, we can show that K_{17} is the first potential counterexample for orientable surfaces.

Main Results

Theorem

Suppose G is a graph drawn on a nonplanar surface Σ with no dependent crossings and G does not contain K_{h+1} as a subgraph. Then $\chi(G) \leq h$.

It is difficult to work with drawings directly.

Converting Drawings to Embeddings

Having an embedding instead of a drawing gives us more tools to work with. Thus, we consider the graph $\eta(G)$ obtained from G by:

- Deleting all crossed edges
- Adding edges to create 4-faces corresponding to the crossings of G
- Triangulating the remaining faces that did not correspond to crossings in G

Step 1: Assign Charge

Assign charges to the vertices and faces of G.

- Usually chosen so that the initial charge is small.

Example: $c(v) = d(v) - 6$ and $c(f) = 2(d(f) - 3)$.

Step 2: Compute Total Charge

Add up the charges to get the total charge of the graph.
In our previous example:

$$\sum_{v \in V(G)} c(v) + \sum_{f \in F(G)} c(f)$$

$$= \sum_{v \in V(G)} (d(v) - 6) + \sum_{f \in F(G)} 2(d(f) - 3) =$$

$$-6\varepsilon$$

Step 3: Discharge

Redistribute charge according to a list of rules.

- Must ensure that no charge is added or lost in this process.
- All discharging happens simultaneously.

For example, we can have each face send an equal portion of its charge to the vertices incident with it.

Step 4: Compute Total Charge

Add up the charges after discharging.

- Use structures in the graph for computation.

We then compare this total charge to the initial charge and see what can be deduced.

Discharging Example

We can use this idea to show that every planar triangulation has a vertex of degree at most 5.
Suppose G is a planar triangulation with $\delta(G) \geq 6$.

- Let $c(v) = d(v) - 6$ and $c(f) = 2(d(f) - 3)$.
- Then $c(G) = -12$.
- For discharging, do nothing.
- Each vertex has $c(v) \geq 0$ since $d(v) \geq 6$.
- This means $c(G) \geq 0$, which is a contradiction.

Results for Embeddings

Given a graph G embedded in a surface, a *cyclic coloring* of G is a coloring in which all vertices incident with the same face receive distinct colors.

Theorem

Suppose G is a graph embedded on a nonplanar surface Σ such that:

1. *each face is either a 3- or 4-face*
2. *the vertices of the 4-faces are pairwise disjoint*
3. *adding a crossing in all 4-faces does not produce K_{h+1} as a subgraph.*

Then G is cyclically h-colorable.

The proof of Heawood's Formula relies on the following:

Theorem

Suppose G is a simple graph that has a 2-cell embedding in a nonplanar surface Σ*. Then G has a vertex of degree at most* $h - 1$*.*

Low Degree Vertices

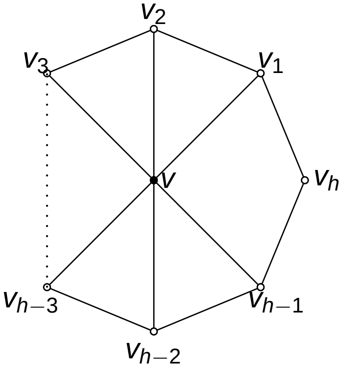
Proposition

No minimal counterexample has a vertex of degree less than h that is not incident with a 4-face.

Proposition

No minimal counterexample has a vertex of degree less than $h - 1$ incident with a 4-face.

$W(\Sigma)$



Proposition

If G is a minimal counterexample containing W, either v_iv_j is an edge or v_i and v_j are incident with the same 4-face for all $i, j \in \{1, 2, \ldots, h-1\}$.

Assign Charge

We charge G as follows:

- $c(v) = d(v) - 6$
- $c(f) = 2(d(f) - 3)$

Thus, $c(G) = -6\varepsilon$.

Bounds on $-\varepsilon$

Using Heawood's formula, we have that

$$h \leq \frac{7 + \sqrt{49 - 24\varepsilon}}{2} < h + 1,$$

which we can rewrite to obtain an upper bound

$$-6\varepsilon < h^2 - 5h - 6$$

Charge of G

W contributes a charge of at least:

$$h^2 - 6h - 5.$$

If $V(G) \neq V(W)$, then the charge of G is at least

$$h^2 - 4h - 12,$$

which exceeds the upper bound of $h^2 - 5h - 6$.
Thus, $V(G) = V(W)$ and, moreover, $G = \eta(K_{h+1})$.

Cyclic Colorings

Given an embedding of a graph G on a nonplanar surface, the *face-width* of G is the smallest number of points that a noncontractible closed curve can have in common with G.

Theorem

Suppose G is a graph embedded on a surface Σ with face-width at least 3 and $\varepsilon < 0$ such that all faces of size greater than 3 are at least distance 3 apart. Then G can be cyclically colored with $n = max\{h+1, \Delta_f(G)+1\}$ colors.

Total Colorings

A *total coloring* is a coloring of the vertices and edges of G such that any pair of incident or adjacent elements receive distinct colors.

Theorem

Suppose G is a graph embedded on a surface Σ with $\varepsilon < 0$ and $\Delta(G) > 4h - 5$. Then $\chi_t(G) = \Delta(G) + 1$.

Total Colorings

Corollary

Suppose G is a graph drawn with no dependent crossings on a surface Σ *with* $\varepsilon < 0$ *and* $\Delta(G) > 4h - 5$*. Then* $\chi_t(G) \leq \Delta(G) + 3$*.*

Theorem

Suppose G is a graph drawn with no dependent crossings on a surface Σ *with* $\varepsilon < 0$ *and* $\Delta(G) > 4h - 4$*. Then* $\chi_t(G) = \Delta(G) + 1$*.*