Pentavalent symmetric bicirculants

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A graph X is said to be *edge-transitive*, if for any two edges $\{u, v\}$ and $\{u', v'\}$ there exist an automorphism $\varphi \in \mathrm{Aut}(X)$ such that $\varphi(\{u, v\}) = \{u', v'\}$.

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A graph X is said to be *edge-transitive*, if for any two edges $\{u, v\}$ and $\{u', v'\}$ there exist an automorphism $\varphi \in \mathrm{Aut}(X)$ such that $\varphi(\{u, v\}) = \{u', v'\}$.
A graph X is said to be *symmetric* (or *arc-transitive*) if for any two pairs of adjacent vertices (u, v) and (u', v'), there exist an automorphism $\varphi \in \mathrm{Aut}(X)$ such that $\varphi(u) = u'$ and $\varphi(v) = v'$.

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The vertices of a bicirculant graph can be labeled by x_i and y_i, $i \in \mathbb{Z}_n$, and its edge set can be partitioned into three subsets

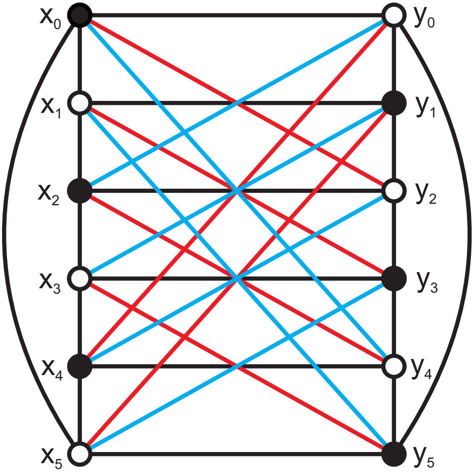
$$\mathcal{L} = \cup_{i\in\mathbb{Z}_n}\{\{x_i, x_{i+l}\} \mid l \in L\},$$

$$\mathcal{M} = \cup_{i\in\mathbb{Z}_n}\{\{x_i, y_{i+m}\} \mid m \in M\},$$

$$\mathcal{R} = \cup_{i\in\mathbb{Z}_n}\{\{y_i, y_{i+r}\} \mid r \in R\},$$

where L, M, R are subsets of $\mathbb{Z}_n$ such that $L = -L$, $R = -R$, $M \neq \emptyset$ and $0 \notin L \cup R$. Such bicirculant is denoted by $BC_n[L, M, R]$.

Example



$$BC_6[\{\pm 1\}, \{0, 2, 4\}\,\{\pm 1\}]$$

Isomorphic bicirculants

Let L, M and R be subsets of $\mathbb{Z}_n$ such that $L = -L$, $R = -R$, $M \neq \emptyset$ and $0 \notin L \cap R$. Then we have:

$$BC_n[L, M, R] \cong BC_n[\lambda L, \lambda M + \mu, \lambda R] \ (\lambda \in \mathbb{Z}_n^*, \ \mu \in \mathbb{Z}_n)$$

with the isomorphism $\phi_{\lambda,\mu}$ given by:

$$\begin{aligned} \phi_{\lambda,\mu}(x_i) &= x_{\lambda i + \mu} \text{ and,} \\ \phi_{\lambda,\mu}(y_i) &= y_{\lambda i}. \end{aligned}$$

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Therefore, we can without loss of generality assume that $0 \in M$. Some graphs may have two (or more) different bicirculant representations, for example:

$$K_{6,6} - 6K_2 \cong BC_6[\{\pm 1\}, \{0, 2, 4\} \{\pm 1\}] \cong BC_6[\,\emptyset, \{0, 1, 2, 3, 4\}, \emptyset\,].$$

Cubic bicirculants I

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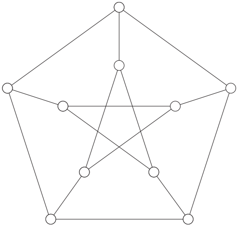
Petersen graph is isomorphic to $GP(5,2)$.



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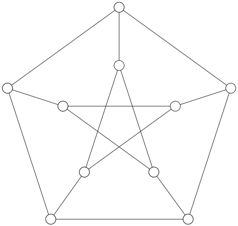

The classification of cubic symmetric bicirculants was later obtained by Marušič and Pisanski.

Cubic bicirculants II

A connected cubic symmetric graph is a bicirculant if and only if it is isomorphic to one of the following graphs:

- the complete graph K_4,
- the complete bipartite graph $K_{3,3}$,
- the seven symmetric generalized Petersen graphs $GP(4,1)$, $GP(5,2)$, $GP(8,3)$, $GP(10,2)$, $GP(10,3)$, $GP(12,5)$, and $GP(24,5)$,
- the Heawood graph $F014A$, and
- a Cayley graph $Cay(D_{2n}, \{b, ba, ba^{r+1}\})$ on a dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1 \rangle$ of order $2n$ with respect to the generating set $\{b, ba, ba^{r+1}\}$, where $n \geq 11$ is odd and $r \in \mathbb{Z}_n^*$ is such that $r^2 + r + 1 \equiv 0 \pmod{n}$.

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The rose window graphs are natural generalization of the generalized Petersen graphs, namely they are bicirculants $BC_n[\{\pm 1\}, \{0, a\}, \{\pm r\}]$. (In the standard notation for rose window graphs, this is equivalent to the graph $R_n(a, r)$).

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The remaining cases were completed by Kovacs, Kuzman, Malnič and Wilson.

Tabačjn graphs

The pentavalent generalization of the generalized Petersen graphs are the so-called *Tabačjn graphs*, that is, a Tabačjn graph is a bicirculant $BC_n[\{\pm 1\}, \{0, a, b\}, \{\pm r\}]$. (In the original notation for Tabačjn graphs, this graph would be denoted by $T(n; a, b; r)$).

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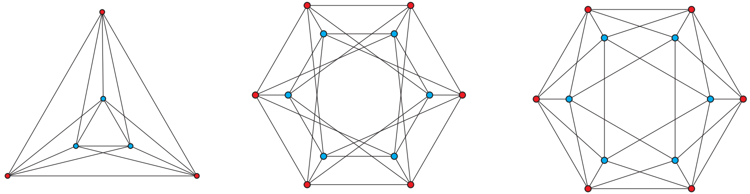
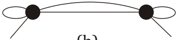
Theorem (Arroyo, Guillot, Hubard, Kutnar, O'Reilly and Šparl, 2013)

The only arc-transitive Tabačjn graphs are the graphs $T(3; 1, 2; 1)$, $T(6; 2, 4; 1)$ *and* $T(6; 1, 5; 2)$.



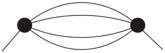
Multigraphs that can occur as quotient multigraphs of pentavalent bicirculants with respect to a $(2, n)$-semiregular automorphism ρ.

(a)



(b)

(c)



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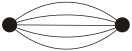

(e)

$|M| = 1$

Theorem (Kovacs, Malnič, Marušič, Miklavič, 2009)

Let $X = BC_n[L, M, R]$ be a pentavalent bicirculant with $|M| = 1$. Then X is not symmetric.

$|M| = 2$

Theorem

Let X be a connected pentavalent symmetric bicirculant $X = BC_n[L, M, R]$ with $|M| = 2$. Then either:

- $n = 6$ *and* $X \cong BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}] \cong K_{6,6} - 6K_2$, *or*
- $n = 8$ *and* $X \cong BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$

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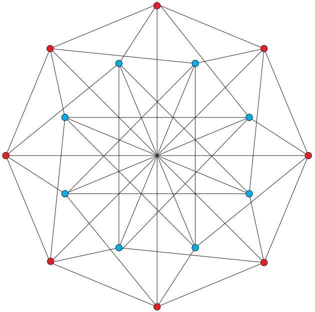

Figure: $BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$

Theorem

Let $X = BC_n[L, M, R]$ be a pentavalent bicirculant with $|M| = 4$. Then X is not symmetric.

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Core-free bicirculants

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Definition

A bicirculant $X = BC_n[L, M, R]$ of order $2n$ is said to be *core-free* if there exists a $(2, n)$-semiregular automorphism $\rho \in \mathrm{Aut}(X)$ giving rise to the prescribed bicirculant structure of X such that the cyclic subgroup $\langle \rho \rangle$ has trivial core in $\mathrm{Aut}(X)$.

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Theorem (Lucchini, 1998)

If H is a cyclic subgroup of a finite group G with $|H| \geq \sqrt{|G|}$, then H contains a non-trivial normal subgroup of G.

Theorem (Guo and Feng, 2012)

Let X be a connected pentavalent (G, s)-transitive graph for some $G \leq Aut(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 5$ and one of the following holds:

(i) *For $s = 1$, $G_v \cong \mathbb{Z}_5, D_{10}$ or D_{20};*
(ii) *For $s = 2$, $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 or S_5;*
(iii) *For $s = 3$, $G_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $S_4 \times S_5$ or $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ with $A_4 \rtimes \mathbb{Z}_2 = S_4$ and $A_5 \rtimes \mathbb{Z}_2 = S_5$;*
(iv) *For $s = 4$, $G_v \cong \mathrm{ASL}(2,4)$, $\mathrm{AGL}(2,4)$, $A\Sigma L(2,4)$ or $A\Gamma L(2,4)$;*
(v) *For $s = 5$, $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma L(2,4)$.*

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Therefore, if $X = BC_n[L,M,R]$ is core-free pentavalent bicirculant, which is $(Aut(X),1)$-transitive, then $n < 40$, and if it is $(Aut(X),2)$-transitive then $n < 240$.

Lemma

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Theorem

The only pentavalent symmetric bicirculants $BC_n[L, M, R]$ *with* $|M| = 3$ *are* $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$, $BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}]$ *and* $BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$. *Moreover, the first two are 2-arc-transitive and the third one is 1-arc-transitive.*

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Proof.

If X is not core-free, then there exist a normal subgroup $N \leq \langle \rho \rangle$. The quotient graph X_N is a core-free pentavalent symmetric bicirculant, and X is a regular $\mathbb{Z}_m$ cover of X_N, where $|N| = m$.

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Lemma (Marušič, 2006)

Let X be a connected pentavalent 2-arc-transitive dihedrant. Then X is core-free if and only if X is isomorphic to the complete bipartite graph minus a matching $K_{6,6} - 6K_2$, or to the points-hyperplanes incidence graph of projective space $B(PG(2,4))$.

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To find all core-free pentavalent symmetric dihedrants, it suffices to check the graphs of order $2n$, where $n < 40$. With use of MAGMA, we obtain that the only such graphs are those mentioned in the previous lemma.

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Theorem

Let X be a connected pentavalent symmetric bipartite dihedrant. Then X is isomorphic to one of the following graphs:

- $K_{6,6} - 6K_2$,
- $BC_{12}[\emptyset, \{0, 1, 2, 4, 9\}, \emptyset]$,
- $BC_{24}[\emptyset, \{0, 1, 3, 11, 20\}, \emptyset]$,
- $B(PG(2, 4))$,
- $Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$ *where* $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1 \rangle$ *and* $r \in \mathbb{Z}_n^*$ *such that* $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{n}$.

Theorem

A connected pentavalent bicirculant $BC_n[L, M, R]$ is symmetric if and only if it is isomorphic to one of the following graphs:

(i) $|M| = 1$: *no graphs;*

(ii) $|M| = 2$: $BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$ *and* $BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$;

(iii) $|M| = 3$: K_6, $K_{6,6} - 6K_2$, $BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$;

(iv) $|M| = 4$: *no graphs*

(v) $|M| = 5$: $K_{6,6} - 6K_2$, $BC_{12}[\,\emptyset, \{0, 1, 2, 4, 9\}, \emptyset\,]$, $BC_{24}[\,\emptyset, \{0, 1, 3, 11, 20\}, \emptyset\,]$, $B(PG(2, 4))$ *or* $Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$ *where* $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$, *and* $r \in \mathbb{Z}_n^*$ *such that* $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{n}$.

Thank you!!!