# Bicircular Matroids with Circuits of Few Sizes

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# Outline

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1. Matroid Designs
2. Results for $|spec(B(G))| \in \{1, 2\}$

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2. Results for $|spec(B(G))| \in \{1, 2\}$
3. Results for $|spec(B(G))| = 3$ where G is 3-connected
   - Without Two Vertex-Disjoint Cycles
   - Coloring the Auxiliary Graph
   - First Result
   - With Two Vertex-Disjoint Cycles
   - Second Result

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- The dual of a matroid with all hyperplanes having the same size is a matroids with all circuits having the same size, called a dual matroid design.

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**Definition**

The circuit-spectrum of a matroid M, denoted by $spec(M)$, is the set of circuit sizes of the matroid.

- We will give the characterization of connected bicircular matroids with circuits of few sizes.

## Definition

Let $G$ be a graph on an edge set $E$. The bicircular matroid of $G$, denoted $B(G)$, has ground set $E$ and circuits being the edge sets of a subdivision of one of the following graphs:

1. Two loops that share a vertex.
2. Two loops with distinct vertices that are joined by an edge.
3. Three edges joining the same pair of vertices.

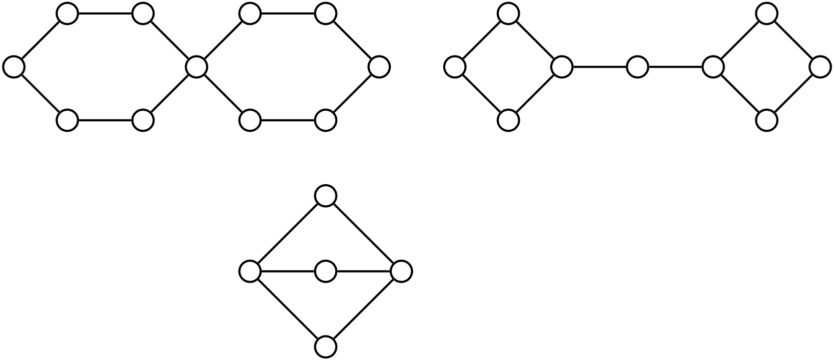

Figure : Types of Bicycles

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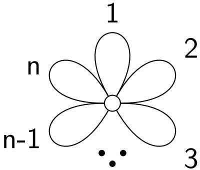
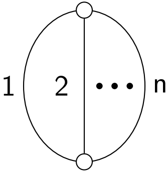
**Theorem (T. Lewis, 2010)**

*Let $M$ be a connected bicircular matroid. For $\eta \geq 2$, $spec(M) = \{\eta\}$ if and only if $M$ is isomorphic to one of the following matroids:*

- *a k-subdivision of $U_{1,n}$, where $\eta = 2k$ and $n \geq 2$*
- *a k-subdivision of $U_{2,n}$, where $\eta = 3k$ and $n \geq 3$*
- *a k-subdivision of $U_{3,5}$, where $\eta = 4k$, or*
- *a k-subdivision of $U_{4,6}$, where $\eta = 5k$.*




$U_{1,n}$

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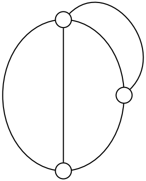
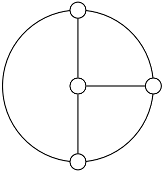


$U_{3,5}$



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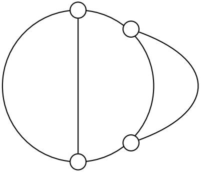

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**Theorem (T. Lewis, 2010)**

*Let $M = B(G)$ be a connected bicircular matroid where $G$ is a subdivision of a 3-connected graph $H$. Then $|spec(M)| = 2$ if and only if $H$ is one of the following graphs:*

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**Theorem (Putnam, L. Sheppardson, T. Lewis, 2013)**

*Let $M = B(G)$ be a connected bicircular matroid where G is not a subdivision of a 3-connected graph H. Then $|spec(M)| = 2$ if and only if H is a restricted subdivision of one of the following graphs:*

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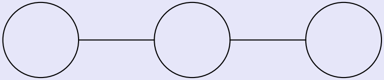

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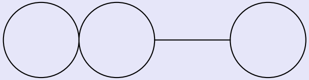

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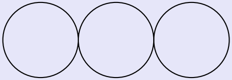

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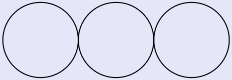
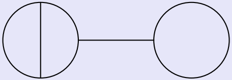


- *A theta with a balloon.*



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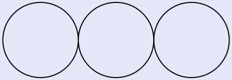

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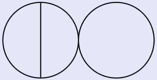

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- *Two equally balanced thetas joined by at most two paths whose endpoints are the same.*

# Outline

We investigate bicircular matroids whose associated graphs are 3-connected.

## Theorem (Wagner, 1985)

*A graph G is a subdivision of a simple 3-connected graph without two vertex-disjoint cycles if and only if G is isomorphic to a subdivision of one of the following graphs: a wheel graph,* $K_5$, $K_5 \setminus e, K_{3,p}$, $K'_{3,p}$, $K''_{3,p}$, $K'''_{3,p}$ *for some* $p \geq 3$.

- We can create an auxiliary graph such that each vertex in the auxiliary graph corresponds to a subdivided edge in the associated graph of the matroid.

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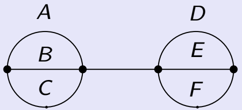
## Associated Graph of a Matroid



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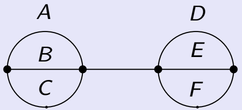
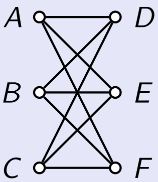



**Auxiliary Graph**



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**Definition (T. Lewis, et. al., 2012)**

Let G be a simple graph, $\varphi : V(G) \rightarrow \mathbb{Z}^+$, and $j \in \mathbb{Z}^+$.

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Let G be a simple graph, $\varphi : V(G) \to \mathbb{Z}^+$, and $j \in \mathbb{Z}^+$.

1. We say that $\varphi$ is a $j$-vertex coloring of G if $|\{\varphi(A) : A \in V(G)\}| = j$.

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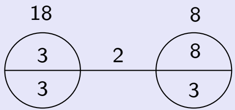
2. We say that $\varphi$ is a $j$-edge coloring of G if $|\{\varphi(A) + \varphi(B) : AB \in E(G)\}| = j$.

## Associated Graph of a Matroid



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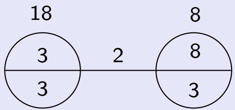
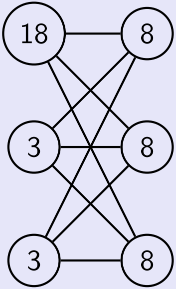



## Auxiliary Graph



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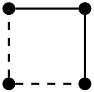

Figure : An alternating four-cycle

## Lemma (Putnam, 2013)

*Let $G$ be a connected graph with a 3-edge coloring $\varphi$.*

- *If a four-cycle is neither edge-monochromatic nor two-edge colored, then one pair of opposite edges have different colors.*

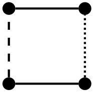

Figure : A 3-edge colored four-cycle

**Theorem (Putnam, 2013)**

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- *a $(2k, k)$-subdivision of $K_5$, $K_5 \setminus e$, $K_{3,3}$, and $W_4$.*

**Lemma**

*Let G be a complete graph on four or more vertices. Then G is 3-edge colored if and only if G is 2-vertex colored.*

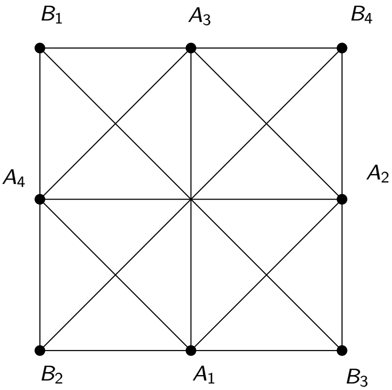


Figure : Auxillary graph of a $W_4$

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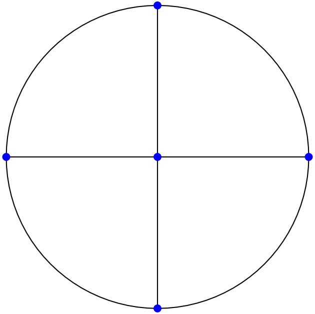

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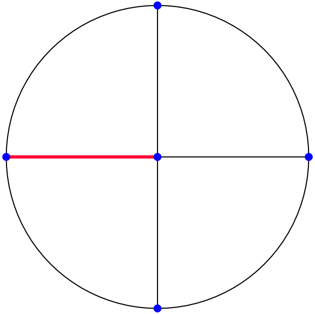

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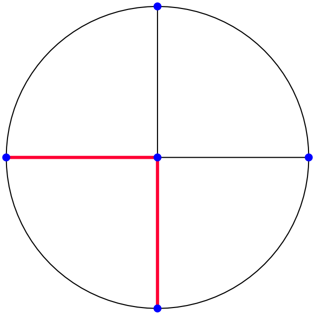

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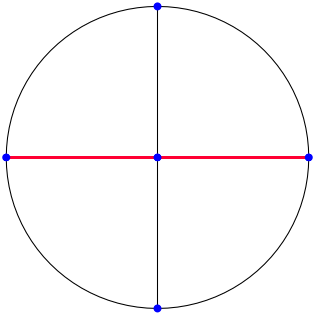

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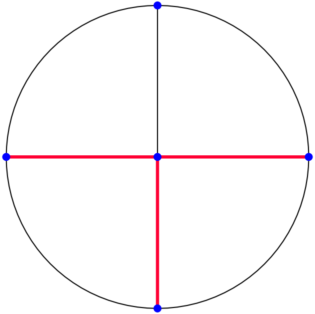

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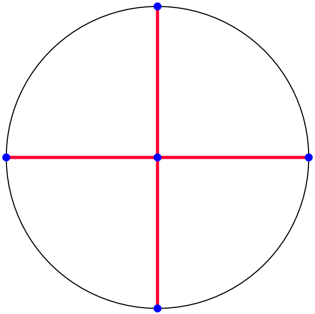

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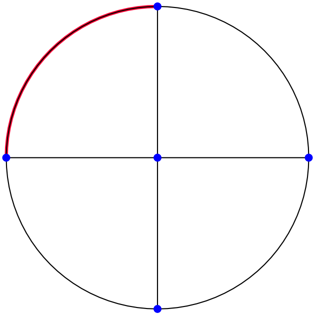

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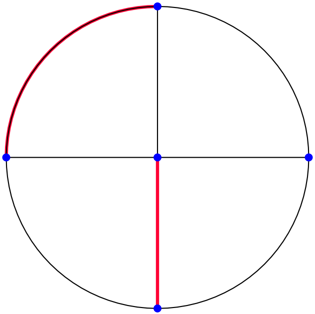

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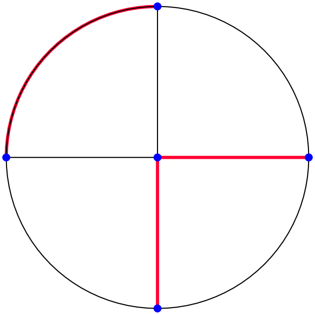
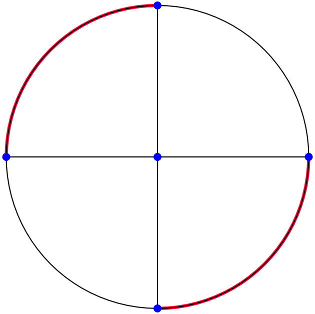


## Lemma (Putnam, 2013)

*Suppose that G is isomorphic to a subdivision of $W_4$. Let $M = B(G)$. Then $|spec(M)| = 3$ if and only if G is isomorphic to one of the following graphs:*



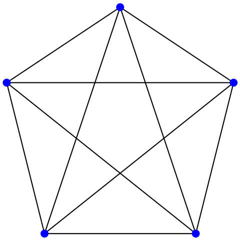


Figure : $K_5$

We consider the case that there are two vertex-disjoint cycles in a 3-connected associated graph.

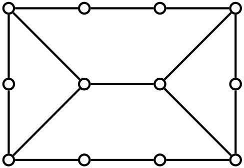


Figure : Two vertex-disjoint cycles

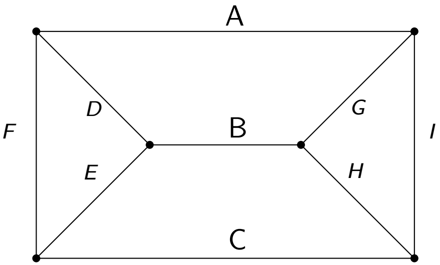



Figure : Prism graph P

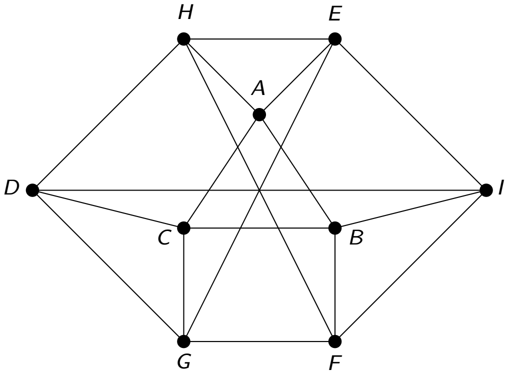
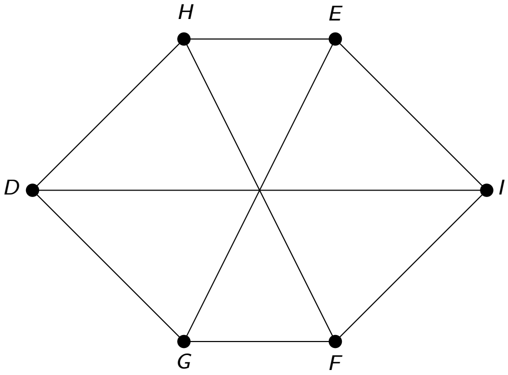



Figure : The auxillary graph K of a prism P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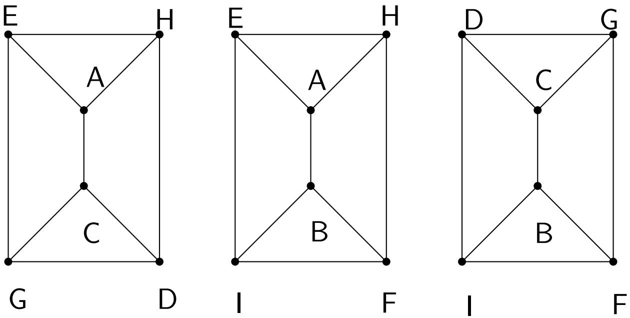



Figure : Three subgraphs of K

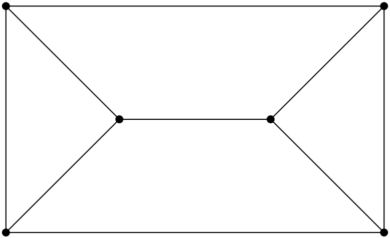
**Theorem (Putnam, 2013)**

*Let G be a subdivision of a simple 3-conneceted graph with two vertex-disjoint cycles. For $M = B(G)$, $|spec(M)| = 3$ if and only if G is isomorphic to one of the following (2k,k)-subdivisions of the prism graph for some $k \in \mathbb{Z}^+$.*

- *Any one edge is $2\beta$-subdivided, then $spec(M) = \{6\beta, 7\beta, 8\beta\}$,*

## Theorem

- *A k-edge-matching is $2\beta$-subdivided for $k \in \{2,3\}$, then $spec(M) = \{k+5\beta, k+6\beta, k+7\beta\}$, Then $|spec(M)| = \{7k, 8k, 9k\}$.*



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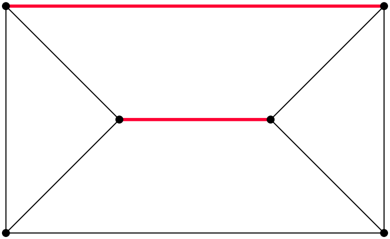

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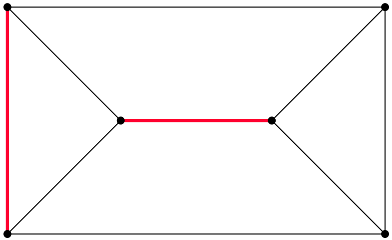

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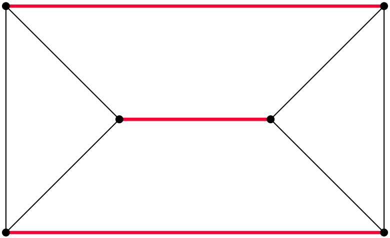

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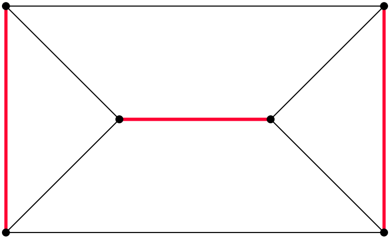
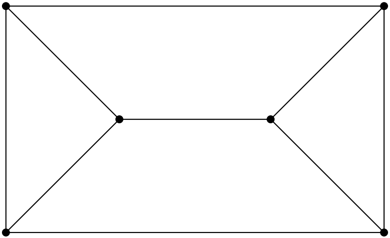


## Theorem

- *A 3-cycle is j-subdivided for* $j \in \{\beta, 2\beta\}$, *then* $spec(M) = \{9\beta, 11\beta, 12\beta\}$ *if* $j = \beta$ *and* $spec(M) = \{7\beta, 9\beta, 10\beta\}$ *if* $j = 2\beta$*;*



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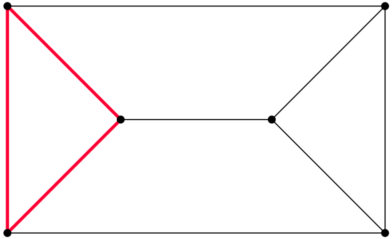

# Outline

- Can we determine the range of the circuit spectrum of a bicircular matroid?

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- Can we find a lower bound on the cardinality of the circuit spectrum of a bicircular matroid given that the associated graph has a large minimum degree?

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Questions?