On binary matroids that guarantee their duals as minors

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What is a binary matroid?

Consider the following matrix viewed over $GF(2)$

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ B = & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

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Let $E = \{1, 2, 3, 4, 5\}$, the set of column labels of B.

Let $\mathcal{I}$ be all the subsets of E whose multisets of corresponding columns are linearly independent.

What is a binary matroid?

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Then $\mathcal{I}$ consists of all subsets of E with at most three elements that do not contain $\{4,5\}$, $\{2,3,4\}$, or $\{2,3,5\}$.

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All binary matroids can be obtained in this way.

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We will call these standard matrix operations.

If a binary matroid M arises from a matrix A, we denote M by $M[A]$ and call A a matrix representation of M.

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The rank $r(M)$ of the matroid M equals the rank of the matrix A.

Connectivity

A matroid $M[A]$ is connected (or 2-connected) if the matrix A cannot be put into the following form using standard matrix operations.

$$\left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$$

Thus the matroid $M[B]$ from before is not connected.

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ B = & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

3-connectivity

A matroid $M[A]$ is 3-connected if the matrix A cannot be put into the following form using standard matrix operations.

$$\left[\begin{array}{cccc|ccc}
 & A_1 & & & & 0 & \\
\hline
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n-1} & \alpha_n \\
\hline
 & 0 & & & & A_2 &
\end{array}\right]$$

Duality

Consider the matrices A and B below.

$$A = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & \cdots & e_r & e_{r+1} & e_{r+2} & \cdots & e_n \end{array} \\ \left[\begin{array}{c|c} I_r & D \end{array} \right] \end{array} \qquad B = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & \cdots & e_r & e_{r+1} & e_{r+2} & \cdots & e_n \end{array} \\ \left[\begin{array}{c|c} D^T & I_{n-r} \end{array} \right] \end{array}$$

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The matroid $M[B]$ is the dual of the matroid $M[A]$.

In general, if M_2 is the dual of M_1, we write $M_1^* = M_2$.

Minors

When deleting an element from a matrix representation, simply remove the specified column: $M[B]\backslash 2$

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ B = & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

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$$B' = \begin{array}{c} \begin{array}{cccc} 1 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

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$M[B]/2$

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1. Make sure that the column is a standard basis vector (if not, then use standard operations).
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$M[B]/2 = M[B'']$

$$B'' = \begin{array}{c} \begin{array}{cccc} 1 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The Question

We want to answer the following question. What binary matroids, if any, guarantee that their duals are present as minors, whenever they themselves are present as minors?

Note that if we want a matroid M to contain some matroid N and its dual N^*, then $\min\{r(M), r(M^*)\} \geq \max\{r(N), r(N^*)\}$.

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We aren't interested in self-dual matroids.

We want to find all binary matroids N that are not self-dual such that if M is a binary matroid for which

$$\min\{r(M), r(M^*)\} \geq \max\{r(N), r(N^*)\},$$

then M has an N-minor if and only if M has an N^*-minor.

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If we insist that N and M are both connected, then the only matroids we get are circuits and parallel classes with four or fewer elements.

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If $|E(N)| < 4$, we get a few trivially 3-connected matroids: $U_{0,1}$, $U_{1,1}$, $U_{1,3}$, $U_{2,3}$.

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We denote the set of 3-connected binary matroids that guarantee the presence of their duals by $\mathcal{N}_3$.

General Strategy

1. For $N \in \mathcal{N}_3$, assume $|E(N)| \geq 4$ and $r(N) > r(N^*)$.

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3. This creates a matroid M that satisfies the obligatory rank constraints, so M must have an N^*-minor.
4. Place $\delta(N)$ elements cleverly to show that, unless N has specific structure, M can't have an N^*-minor.

The Fano Plane

$$F = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array} \quad F^* = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The matroids $M[F] = F_7$ and $M[F^*] = F_7^*$, are the Fano plane and its dual.

The Fano Plane

Using our strategy, we begin with F_7^* and consider adding a column to the matrix F^*.

$$
F^* = \begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

Seymour (1985)

$$G = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$H = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

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0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{array}
$$

Both $M[G]/4$ and $M[H]/4$ are isomorphic to F_7.

Main Result

Theorem (T.)

Let N be a 3-connected binary matroid that is not self-dual such that $|E(N)| \geq 4$, and let M be a 3-connected binary matroid for which $\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}$. The following are equivalent:

(i) *M has an N-minor if and only if M has an N^*-minor.*

(ii) *The matroid N is F_7, F_7^*.*

Proof Sketch

Take $N \in \mathcal{N}_3$ and show:

1. N has a triad, but no triangles.

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Proof Sketch

Take $N \in \mathcal{N}_3$ and show:

1. N has a triad, but no triangles.
2. $\delta(N) = r(N) - r(N^*) = 1$.
3. N has multiple triads.
4. N has no disjoint triads, so N has a pair of intersecting triads.

Proof Sketch

Let T_1^* and T_2^* be the intersecting triads, and let $T_1^* \cap T_2^* = \{x\}$.

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The complement of a triad in N is a hyperplane of N. We let H_1 and H_2 be hyperplanes that are complementary to T_1^* and T_2^*.

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Consider adding an element e to the flat $H_1 \cap H_2$, to get a matroid M.

Proof Sketch

Using this construction, we must contract x to obtain an N^*-minor from M.

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Add e cleverly in different spots to get two constructions of N^* that have differing numbers of triangles.

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This strategy can be used to show that $r(N) \leq 5$.

Spikes

The matroid F_7 is the rank-3 binary spike. In general, the rank-r binary spike Z_r is the matroid associated with the binary matrix below.

$$\left[\begin{array}{c|ccccc|c} & 0 & 1 & 1 & \cdots & 1 & 1 \\ & 1 & 0 & 1 & \cdots & 1 & 1 \\ I_r & 1 & 1 & 0 & \cdots & 1 & 1 \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & 1 & 1 & 1 & \cdots & 0 & 1 \end{array}\right]$$

Proof Sketch

Theorem (Oxley)

Let M be a binary matroid with $r(M) \geq 3$. Then M is 3-connected and has no $M(\mathcal{W}_4)$-minor if and only if $M \cong Z_r, Z_r^*$, or $Z_r \backslash e$, for some $r \geq 3$ and any $e \in Z_r$.

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Thus either N is a spike or the dual of a spike, or N has a four-wheel minor.

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If N is a spike or the dual of a spike, then $N \in \{F_7, F_7^*, Z_4, Z_4^*\}$.

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Thus either N is a spike or the dual of a spike, or N has a four-wheel minor.

If N is a spike or the dual of a spike, then $N \in \{F_7, F_7^*, Z_4, Z_4^*\}$.

Assume $N \in \{Z_4, Z_4^*\}$.

Proof Sketch

$$S = \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

This matrix is a representation of Z_4.

Proof Sketch

Extend Z_4 by two elements to get M_0 (add two columns to S).

$$\begin{array}{cc} & \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ S' = & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Proof Sketch

Now coextend M_0 by a single element e to get M.

$$S'' = \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & \color{orange}{e} & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \left[\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

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$M[S'']$ satisfies the obligatory rank constraints and has Z_4, but not Z_4^*, as a minor.

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Since $r(N) \leq 5$, we know N is obtained from the four-wheel by a single-element coextension.

There is only one coextension, up to isomorphism, that is 3-connected and has no triangles, but it isn't in $\mathcal{N}_3$.

Theorem (T.)

Let N be a 3-connected binary matroid that is not self-dual such that $|E(N)| \geq 4$, and let M be a 3-connected binary matroid for which $\min\{r(M), r^*(M)\} \geq \max\{r(N), r^*(N)\}$. The following are equivalent:

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