# Decomposition of Cubic graphs on the Torus and Klein-bottle

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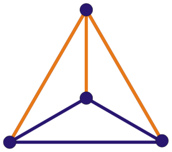
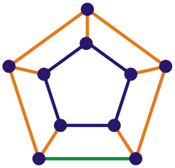
# Definitions

The **decomposition** of a graph G consists of edge-disjoint subgraphs whose union is G





# Previous Results

- (Balogh, Kochol, Pluhar & X. Yu, 2005) Every plane graph has a decomposition into forests and one of them has maximum degree 8.
- (Goncalves, 2009) Every plane graph can be decomposed into 3 forests and one of them has maximum degree 4.
- (Kleitman, 2006) Every plane graph with girth at least 6 has a decomposition into a forest and a family of paths.
- (He, Hou, Lih, Shao, Wang, Zhu, 2002; Borodin, Kostochka, Sheikh, G. Yu, 2009; Wang, Zhang, 2011) Every plane graph with large girth (at least 8) has a decomposition into a forest and a matching.
- (Kim, Kostochka, West, Wu & Zhu 2013) Every $(k, d)$-sparse graph has a decomposition into a $k$ forests and one subgraph with maximum degree $d$.

# Previous Results

What about decompositions of cubic graphs into a spanning tree and other subgraphs?

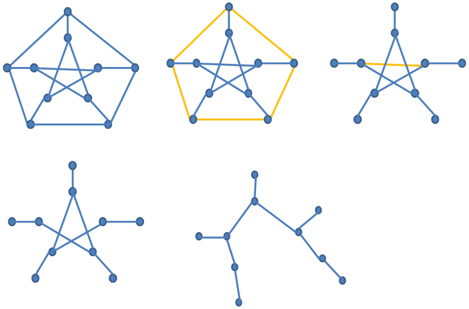
- (Malkevitch,1979)
  Which 3-connected plane cubic graphs have a decomposition into a spanning tree and a family of cycles?
- (Lemke, 1988) It is NP-complete to determine whether a given cubic graph has the decompstion or not
- It reamins NP-complete for 3-connected plane cubic graphs

# Previous Results

- (Hoffman-Ostenhof,2009) Every cubic graph has a decomposition into a spanning tree, a matching, and a family of cycles
- (Kostochka, 2009) The conjecture is verified for many graphs including the Pertersen graph.
- (Ozeki and Ye, 2014+) The conjecture is true for 3-connected cubic graphs on the sphere and projective plane.
- (Bachstein and Ye, 2015+) Every 3-connected cubic graph on the torus and Klein-bottle has a decomposition into a spanning tree T, a matching, and a family of cycles.

# Example

(Hoffman-Ostenhof,2009) Every cubic graph has a decomposition into a spanning tree, a matching, and a family of cycles



# Decomposing Cubic Graphs

Every 3-connected cubic graph on the torus and Klein-bottle has a decomposition into a spanning tree T, a matching, and a family of cycles.

**Outline of Proof**:

- Assume there exists a non-contractible, non-separable cycle $C$ such that $G - C$ is a cylinder
- Cylinder is homeomorphic to an annulus.

# Decomposing Cubic Graphs

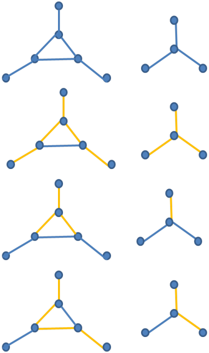
**Technical Lemma**: Let G be a connected plane graph with maximum degree at most 3. Suppose that

- (1) All cut edges appear on the two boundaries, $\partial_1 G$ and $\partial_2 G$
- (2)for all 2-edge cuts S, both edges in S are contained in $\partial_1 \cup \partial_2$
- (3)all edges incident with a vertex of degree at most 2 are either cut-edges of G or on $\partial_1 \cap \partial_2$
- (4) If $\partial_1 \cap \partial_2$ contains degree-2 vertices then there exists a pair of degree-2 vertices on $\partial_1 \cap \partial_2$ which is separated by at least two consecutive leaf-attachments or a block.

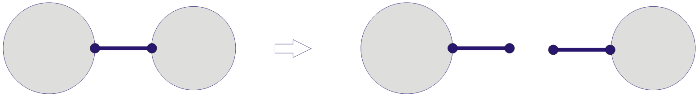
Then G can be decomposed into a spanning tree T, a matching M, and a family of facial cycles of G.

# Decomposing Cubic Graphs

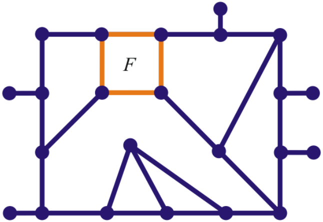
Note that G is triangle free



There exists no cut-edge $e$ in $G$ such that $G - e$ has two components with at least two vertices

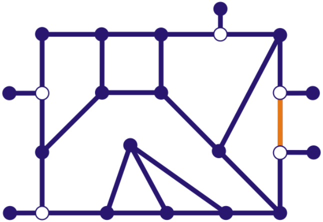


If $F$ is a second outer facial cycle of $G$, then $F$ contains either a leaf-attachment or a cyclic 2-edge-cut of $G$.

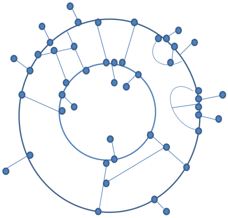


# Decomposing Cubic Graphs

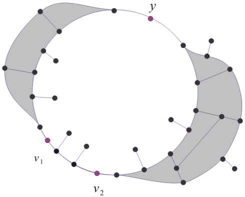
A **leaf-attachment** is a vertex of degree-3 incident with a leaf
No two leaf-attachments are adjacent



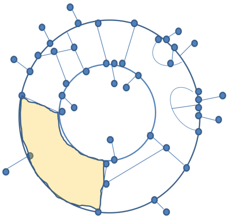
**Case 1**: $\partial_1 \cap \partial_2 = \emptyset$



**Case 2**: $\partial_1 \cap \partial_2 \neq \emptyset$



**Case 1**: $\partial_1 \cap \partial_2 = \emptyset$
(1) If no face touches both boundaries, then start at one boundary and use Ozeki and Ye, 2014 result
(2) If at least one face is adjacent to both boundaries



# Decomposing Cubic Graphs

**Case 1**: $\partial_1 \cap \partial_2 = \emptyset$

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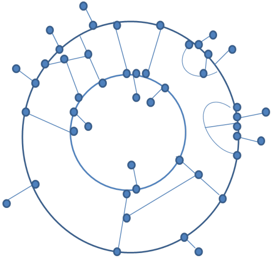
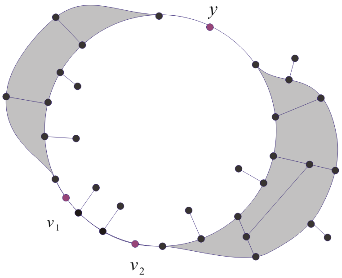
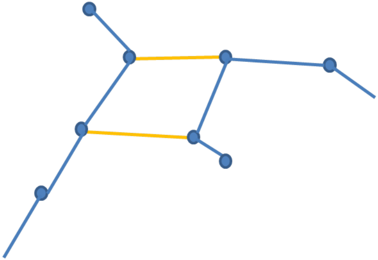


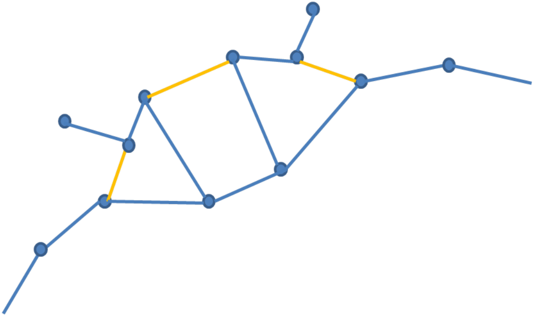
**Case 2**: $\partial_1 \cap \partial_2 \neq \emptyset$



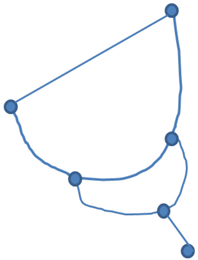
**Case 2**: $\partial_1 \cap \partial_2 \neq \emptyset$
Since $G$ is triangle free, the first case is



**Case 2**: $\partial_1 \cap \partial_2 \neq \emptyset$
The first non-trivial case



**Case 2**: $\partial_1 \cap \partial_2 \neq \emptyset$
We also allow stacked faces



**Thank You**