Minimal Symmetric Differences of lines in Projective Planes

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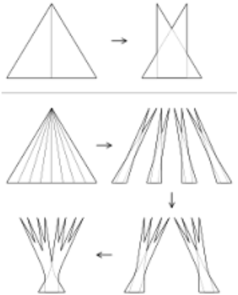
The Kakeya problem

What is the smallest set in the plane that contains a unit line segment in every direction?

Theorem (Besicovich, 1963)

The set can have arbitrarily small area.



The Kakeya problem in $\mathbb{F}_q^n$

What is the smallest set S in the affine space $\mathbb{F}_q^n$ that contains a line in every direction?

Theorem (Dvir, 2008)

$|S| \geq \binom{q^n+n-1}{n}$.

This proves the Finite field Kakeya conjecture (Wolfe, 1999) that $|S| \geq c_n|\mathbb{F}_q^n|$.

In 2 dimensions, $|S| \geq \frac{q(q+1)}{2}$.
Best known upper bound $|S| \leq \frac{q(q+1)}{2} + \frac{5q}{14} + O(1)$ (Cooper, 2006)

A generalization

Instead of insisting that all directions are represented, what if we just insist that we have r distinct lines. For simplicity we shall now work in projective space $PG(2,q)$.

Let $\mathcal{P}$ be the set of points, $\mathcal{L}$ the set of lines in $PG(2,q)$, and $N = |\mathcal{P}| = |\mathcal{L}| = q^2 + q + 1$.

Question

What is $$g(r) = \min_{R \subseteq \mathcal{L},\ |R|=r} \Big| \bigcup_{\ell \in R} \ell \Big| ?$$

Note that by duality between points and lines

$$g(r) = \min_{S \subseteq \mathcal{P},\ |S|=r} \Big| \{\ell \in \mathcal{L} : \ell \cap S \neq \emptyset\} \Big|.$$

Unions of lines

Clearly $g(r) \geq r(q+1) - \binom{r}{2}$ as each pair of lines intersects in one point, and the smallest union occurs when all these intersections are distinct.

In the dual viewpoint we have equality $g(s) = s(q+1) - \binom{s}{2}$ iff no three points of S are co-linear.

Lemma

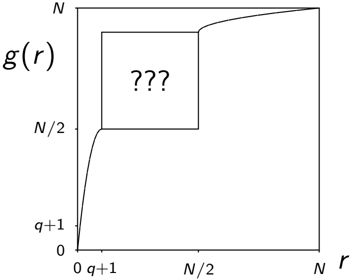
$g(r) = r(q+1) - \binom{r}{2}$ *for* $r \leq q+1$.

Proof. Use a subset S of size r of a conic in $PG(2,q)$. □

Unions of lines

$$g(r) \leq s \quad \Leftrightarrow \qquad g(N-s) \leq N-r \quad \Leftrightarrow$$

There exists a point set of size $N-s$ non-incident to a line set of size r. Hence



Question

What is $g(r)$ for $q+1 < r < (q+1)(q+2)/2$?

Symmetric differences

Instead of unions, what happens if we take symmetric differences?

Consider subsets of $\mathcal{P}$ as a binary vector in $\mathbb{F}_2^{\mathcal{P}}$.
From now on we shall always assume q is odd.

Define

$$f(r) = \min_{R \subseteq \mathcal{L},\ |R|=r} \Big| \sum_{\ell \in R} \ell \Big|$$

to be the size of the smallest symmetric difference between r distinct lines of $PG(2, q)$.

Symmetric differences

Define for $R \subseteq \mathcal{L}$, $S \subseteq \mathcal{P}$,

$$\mathcal{P}^o(R) = \sum_{\ell \in R} \ell = \{p \in \mathcal{P} : p \text{ lies in an odd number of } \ell \in R\},$$
$$\mathcal{L}^o(S) = \{\ell \in \mathcal{L} : |\ell \cap S| \text{ is odd}\}.$$

Then

- $\mathcal{P}^o \colon \mathbb{F}_2^{\mathcal{L}} \to \mathbb{F}_2^{\mathcal{P}}$ and $\mathcal{L}^o \colon \mathbb{F}_2^{\mathcal{P}} \to \mathbb{F}_2^{\mathcal{L}}$ are both linear maps.
- $\ker \mathcal{P}^o = \{\emptyset, \mathcal{L}\}$, $\ker \mathcal{L}^o = \{\emptyset, \mathcal{P}\}$.
- $|\mathcal{P}^o(R)|$ and $|\mathcal{L}^o(S)|$ are always even.
- $\mathcal{P}^o$ and $\mathcal{L}^o$ are inverse isomorphisms between the even weight subspaces of $\mathbb{F}_2^{\mathcal{P}}$ and $\mathbb{F}_2^{\mathcal{L}}$.

Also

$$f(r) = \min_{R \subseteq \mathcal{L},\ |R|=r} |\mathcal{P}^o(R)| = \min_{S \subseteq \mathcal{P},\ |S|=r} |\mathcal{L}^o(S)|.$$

Some simple observations

Lemma

$f(r) = f(N - r)$

Proof. Each point lies in $q + 1$ lines, and $q + 1$ is even, so $\sum_{\ell \in \mathcal{L}} \ell = 0$.
Thus $|\mathcal{P}^o(R)| = |\mathcal{P}^o(\mathcal{L} \setminus R)|$ and so $f(r) = f(N - r)$. □

Some simple observations

Lemma

- $r(q+2-r) \leq f(r) \leq rq + (r \bmod 1)$,
- $f(r) = r(q+2-r)$ *for* $0 \leq r \leq q+1$.

Proof. To minimize the symmetric difference between a set of lines, one would like all intersection points between lines to be distinct. Then $|\mathcal{P}^o(R)| = r(q+2-r)$. This can be obtained by taking the dual of r points on a conic if $r \leq q+1$.

It is clear that $f(1) = q+1$ and $f(2) = 2q$, and $f(x+y) \leq f(x) + f(y)$. Hence $f(r) \leq rq + (r \bmod 1)$. □

Some simple observations

Lemma

$f(r) \equiv r(q+2-r) \bmod 4.$

Proof. Consider adding the rth line. It must meet all previous $r-1$ lines, so the number of intersection points where it meets an odd number of previous lines is $x \equiv r-1 \bmod 2$. But the symmetric difference then increases by $q+1-2x \equiv q+3-2r \bmod 4$. Thus the symmetric difference is the same mod 4 as if all intersection points between pairs of lines are distinct. □

Some observations

Lemma

$|f(r+1) - f(r)| \le q - 1$ *for* $0 < r < N-1$.

Proof. One can always add a line that meets $\mathcal{P}^o(R)$ when $R \neq \emptyset, \mathcal{L}$. Thus $f(r+1) \le f(r) + q - 1$.
The reverse inequality follows as $f(r) = f(N-r)$. □

There are in fact several values of r for which $|f(r+1) - f(r)| = q - 1$.

The middle range

For almost all values $Cq^{3/2} < r < N - Cq^{3/2}$, it is possible to calculate $f(r)$ exactly. (However, there does not seem to be a nice way of describing the answer). In particular, for these vales of r, $f(r)$ is quite small.

Theorem

$f(r) \leq q$ for $Cq^{3/2} < r < N - Cq^{3/2}$.

The middle range

Fix an set S of points of even size. Then if $R = \mathcal{L}^o(S)$ we have $S = \mathcal{P}^o(R)$. As $f(r)$ is always even, determining $f(r)$ for even r is equivalent to the following:

Find the smallest even sized set S such that $|\mathcal{L}^o(S)| = r$.

For odd r we have $f(r) = f(N - r)$ and $N - r$ is even.

Clique decompositions

Given S, we can use the lines of the projective plane to edge-decompose the complete graph K_S into cliques $K_{\ell \cap S}$.
Indeed, each edge of K_S lines in a unique line $\ell \in \mathcal{L}$ and this line joins all pairs of points in $\ell \cap S$.

List the lines of $\mathcal{L}$ as $\ell_1, \ldots, \ell_N$ and define $s_i = |\ell_i \cap S|$.
Let Π be an edge-decomposition of K_S into cliques of size s_i. Define

$$M(\Pi) = \sum \left\lfloor \frac{s_i}{2} \right\rfloor$$

Lemma

If $|S| = s$ is even, and Π is the clique decomposition corresponding to S. Then $|\mathcal{L}^o(S)| = s(q+1) - 2M(\Pi)$.

Clique decompositions

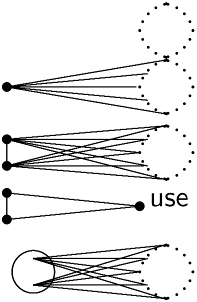
Hence for a given size s of S, it is enough to:

- Determine the possible values of $M(\Pi)$ when Π is an arbitrary clique decomposition of K_s.
- Determine which of these clique decompositions can be realized in the projective plane.

In practice, for s not too close to 0 or $q+1$, one gets a solid range of possible values for $M(\Pi)$, subject to parity, from about $s+O(\sqrt{s})$ to $\gg s$. We also get a few explicitly determined values from s to $s+O(\sqrt{s})$. From these it is easy to determine the minimum $|S|$ for which $|\mathcal{L}^o(S)|=r$ is solvable when r is not too close to 0 or N.

Clique decompositions

Π	$M(\Pi)$	Clique decomposition
s	$\lfloor \frac{s}{2} \rfloor$	
$s-1, 2, \ldots, 2$	$\lfloor \frac{s-1}{2} \rfloor + s - 1$	
$s-2, 2, 2, \ldots, 2$	$\lfloor \frac{s-2}{2} \rfloor + 2(s-2) + 1$	
$s-2, 3, 2, \ldots, 2$	$\lfloor \frac{s-2}{2} \rfloor + 2(s-2) - 1$	use as triangle
$s-i, 2, \ldots, 2$	$\lfloor \frac{s-i}{2} \rfloor + i(s-i) + \binom{i}{2}$	
$s-i, 3, \ldots, 2$	$\lfloor \frac{s-i}{2} \rfloor + i(s-i) - \binom{i}{2}$	varies in steps of 2



As $s_1 = s - i$ decreases, a range of values (in steps of 2) is possible. For $i \gg \sqrt{s}$ these ranges overlap and give a solid range of possible $M(\Pi)$.

Clique decompositions

Definition

Π is simple if all but one clique is either an edge or a triangle.

Theorem

If there exists a clique decomposition of K_s with $M(\Pi) < \frac{1}{4}s(\sqrt{4s-3}-1)$ then there exists a simple clique decomposition Π' with $M(\Pi') = M(\Pi)$.

As the interesting $M(\Pi)$ are $O(s)$, we can reduce to the case of simple clique decompositions.

Realizing decompositions

Suppose we have a (simple) clique decomposition of K_s, can we realize is with a set S of size s in $PG(2, q)$?

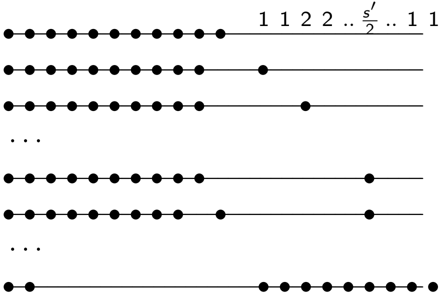
Construction:

Put s_1 points on a line at infinity ℓ_∞, say $T \subseteq \ell_\infty$. Put the other $s' = s - s_1$ points C on the conic $y = x^2$, at $(0, 0)$, $(1, 1)$, $(2, 4)$,. . . .

Then ℓ_∞ induces the clique K_{s_1} and all lines through the remaining points induce either K_2s or K_3s. The number of K_3s is the number of lines through two points of C that meet T.

Realizing decompositions

Note that there is 1 line through points of C with slope 1, 1 with slope 2, 2 with slope 3, 2 with slope 4, ...



Theorem

If $0 \leq s \leq q+1$ and $s_1 \geq \max\{(2s-3)/3, (2s-3)-(q+1)\}$ then any simple decomposition Π of K_s can be realized by a set of points in $PG(2,q)$.

Calculating $f(r)$

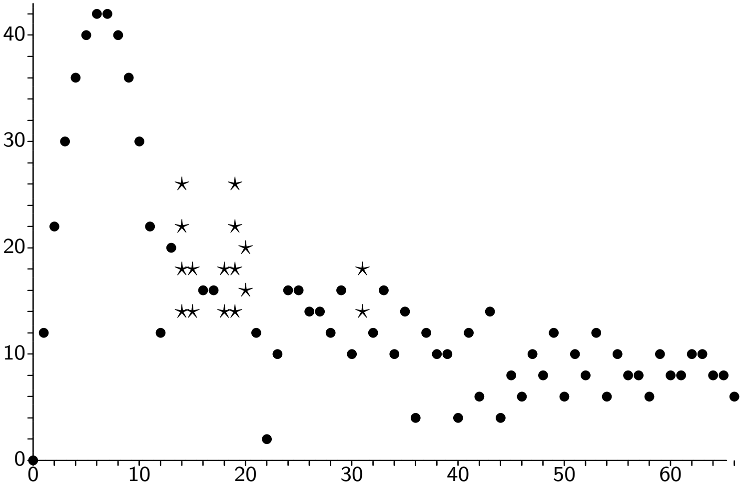
If r is odd, calculate $f(N-r)$ instead.

Loop through even s with $qs \geq r$.

> If there is a simple clique decompositions of K_s with $r = s(q+1) - 2M(\Pi)$ and it can be realized in $PG(2,q)$, return s.
>
> Otherwise, if there is a simple clique decompositions of K_s with $r = s(q+1) - 2M(\Pi)$, return "undetermined"
>
> If not every clique decomposition is equivalent to a simple one, return "undetermined".

Repeat

Numerical results $q = 11$



Other results

Theorem

The maximum of $f(r)$ *occurs at* $r = (q+1)/2, (q+3)/2, N-(q+1)/2, N-(q+3)/2.$

Theorem

$f(2q-1) = q+1$, $f(2q) = 2$, $f(2q+1) = q-1.$

Theorem

$3(q+1)/2 \le f(q+2) \le 2q-2.$

Conjecture

$f(q+2) = 2q-2.$

$f(q+2)$

Theorem (Bichara, Korchmáros, 1980)

Let R be a set of $q+2$ lines in $\mathcal{L}$, then there are at most 2 lines without triple points.

Proof. Assume there are 3 lines without triple points. Wlog they are $x=0$, $y=0$, and the line at infinity. But then each other point on these lines intersects exactly one of the remaining q lines of R.
The remaining lines are $a_i(x-b_i)$ with $\{a_i\}=\{b_i\}=\{a_ib_i\}=\mathbb{F}_q^\times$. But $\prod_{x\in\mathbb{F}_q^\times} x=-1$ so $-1=\prod a_ib_i=(\prod a_i)(\prod b_i)=(-1)(-1)$, a contradiction. □

$f(q+2)$

Theorem (Jamison (1977), Brouwer and Schrijver (1978))

Any blocking set in $\mathcal{P}$ contains at least $2q-1$ points.

Proof. Let B be a blocking set. Wlog $(0,0) \in B$. Consider
$$f(x,y) = \prod_{(a_i,b_i)\in B\setminus\{(0,0)\}} (a_i x + b_i y - 1).$$
Then for each $(a,b) \neq (0,0)$ the line $ua + vb - 1$ meets $B \setminus \{(0,0)\}$, so $f(a,b) = 0$. But $f(0,0) = \pm 1$.
Write $f(x,y) \equiv g(x,y) \bmod (x^q - x, y^q - y)$, with $\deg_x g, \deg_y g < q$. Then xg is identically zero on $\mathbb{F}_q^2$. Thus $xg \in (x^q - x, y^q - y)$. But $\deg_y g < q$, so $x^q - x \mid xg$ and so $x^{q-1} - 1 \mid g$. Similarly $y^{q-1} - 1 \mid g$. But then $(x^{q-1} - 1)(y^{q-1} - 1) \mid g$, so $\deg_{\text{total}} f \geq \deg_{\text{total}} g \geq 2q - 2$. Hence $|B \setminus \{(0,0)\}| \geq 2q - 2$ and $|B| \geq 2q - 1$. □

$f(q+2)$

Lemma

Suppose $|R| = q+2$ and at least one line of R has no triple points. Then $|\mathcal{P}^o(R)| \geq 2q-2$.

Proof. Assume the line at infinity ℓ_∞ lies in R and has no triple points. Then in the Affine plane, no two finite lines of R are parallel. As there are $q+1$ finite lines, every line in $\mathbb{F}_q^2$ must be parallel to a unique line of R.
Claim: $\mathcal{P}^o(R) \cap \mathbb{F}_q^2$ blocks all lines except those of R that have no triple point.
Proof. If $\ell \notin R$ then ℓ meets an odd number $(q+1-1)$ of finite lines of R and so has an odd point.
If $\ell \in R$ and R has a triple point, then not all points on ℓ intersect another element of R. Such a point is single, so odd.
Finally, we can assume there are at most 1 finite line of R without triple points and this can be blocked by adding a single point to $\mathcal{P}^o(R)$. Thus $|\mathcal{P}^o(R)| + 1 \geq 2q-1$. □

$f(q+2)$

If every line of R has a triple point, the best we can do is $|\mathcal{P}^o(R)| \geq \frac{3}{2}(q+1)$.

We know $f(q+2) = 2q-2$ for $q \leq 13$.

Other constructions

For $r \approx 3q/2$, $f(r)$ is quite small due to the following construction (due to J. di Paola):

Let $Q^+ \subseteq \mathbb{F}_q$ be the set of non-zero quadratic residues, and $Q^- \subseteq \mathbb{F}_q$ the set of quadratic non-residues. Define

$$Q = \{[x{:}0{:}1] : x \in Q^+\} \cup \{[1{:}x{:}0] : x \in Q^+\} \cup \{[0{:}1{:}x] : -x \in Q^-\}.$$

Then $|\mathcal{L}^o(Q)| = |Q| = 3(q-1)/2$, so $f(3(q-1)/2) \leq 3(q-1)/2$.

Similar constructions show that $f(r)$ is small near $2q, 5q/2, 3q, 7q/2, \ldots$

Open problems

- Calculating or just estimating $g(r)$ for $q + 1 < r < (q + 1)(q + 2)/2$.
- Proving $f(q + 2) = 2q - 2$.
- Determining at what point $f(r)$ becomes $O(q)$ as r increases.
- Determining a (polynomial time) algorithm for calculating $f(r)$ for all r.
- Non-Desarguesian planes? (The $f(r)$ is affected by the structure of the plane.)

The End