An Erdős-Szekeres result for set partitions

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Outline

- The Erdős-Szekeres result
- Analogous definitions and terminology for set partitions
- Our result
- An immediate corollary on sequences
- A refinement
- A conjecture refuted
- Future directions
- References

An early Ramsey-theoretic result

Theorem: [Erdős-Szekeres (1935)] Every $(n^2 - 2n + 2)$-sequence of distinct numbers must have a monotonic n-subsequence. This is the least integer with this property.

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Example: $n = 4$. The 9-sequence 678345012 has no monotonic 4-subsequence, but every 10-sequence does, e.g., 5841629073.

An analogous result for set partitions

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- An orderly property (*E-S: monotonicity*)

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Some things we will need:

- A measure of the size of a set partition (*E-S: sequence length*)
- A definition of subpartition (*E-S: subsequence*)
- An orderly property (*E-S: monotonicity*)
- A minimum $f(n)$ such that set partitions of this size have an orderly subpartition of size n (*E-S:* $n^2 - 2n + 2$)

The weight of a set partition

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Example: $\mathrm{wt}(156/2/378/4) = 8$.

Subpartitions

If $\pi = B_1/\cdots/B_k$ is a partition on $S \supseteq T$, let

$$\pi|_T = \{B_i \cap T \neq \emptyset\}.$$

μ is a *subpartition* of π if $\mu = \pi|_T$ for some $T \subseteq S$.

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Example: 5/78/4 is a subpartition of 156/2/378/4.

An orderly property

A set partition is *free* if it has no subpartition of the form ac/b, where $a < b < c$.

Alternatively, a set partition is free if it can be written $B_1/\cdots/B_k$ so that every element of B_i is less than every element of B_{i+1} for $1 \leq i < k$.

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Example: $156/2/378/4$ is not free, but $5/78/4 = 4/5/78$ is.

A minimum function

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Proof: Upper bound: induction on n using the fact that $f(n) = \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ satisfies $f(n) = f(n-2) + n$.
Lower bound: modify of a partition with $\left\lceil \frac{n+1}{2} \right\rceil$ blocks of size $\left\lfloor \frac{n+1}{2} \right\rfloor$. $\square$

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$\left\{ \left\lfloor \frac{n^2}{4} \right\rfloor \right\}_0^\infty = 0, 0, 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, \ldots$ is A006260 in the OEIS. Dozens of items are counted by this sequence.

A correspondence between set partitions and sequences

An (ordered) set partition $B_1/\cdots/B_k$ on $\{s_1 < \cdots < s_m\}$ corresponds to a surjective sequence $t_1 \cdots t_m$ on $\{1, \ldots, k\}$ via $t_r = j$ when $s_r \in B_j$.

Example: $5/78/4 \to 3122$ and $156/2/378/4 \to 12341133$.

A different orderly property for sequences

A sequence $t_1 \dots t_m$ is *separated* if $i < j$ and $t_i = t_j$ implies $t_r = t_i$ for all $i \le r \le j$.

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Example: 3122 is separated, while 12341133 is not.

Our result for sequences

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Example: $n = 5$. The 9-sequence

$$18/26/3479/5 \to 123342313$$

has a separated 5-subsequence, but the 8-sequence

$$147/258/36 \to 12312312$$

does not.

A refinement

Let $$M(n,k) = \begin{cases} k(n-k+1) & 1 \le k \le (n+2)/2 \\ \left\lfloor \frac{(n+1)^2}{4} \right\rfloor & (n+2)/2 < k < n \\ k & k \ge n \end{cases}.$$

Theorem: [G. & Sheard] Every $M(n,k)$-partition with exactly k blocks has a free n-subpartition. This is the least integer with this property.

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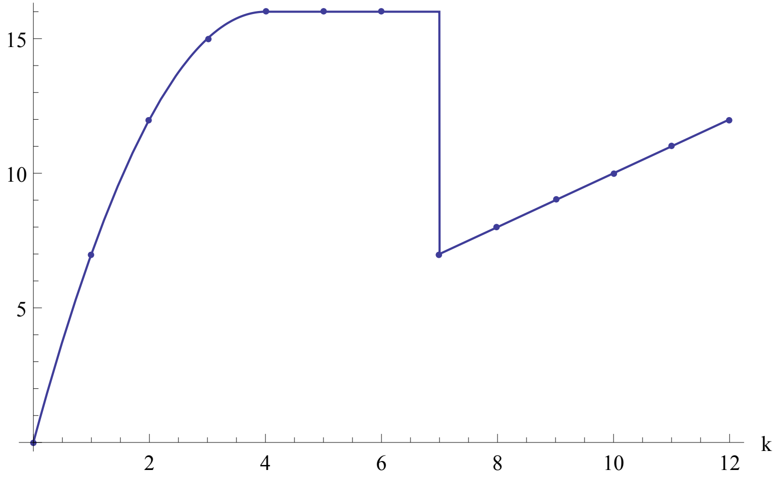
Theorem: [G. & Sheard] Every surjective $M(n,k)$-sequence on $\{1,\ldots,k\}$ has a separated n-subsequence. This is the least integer with this property.

$M(7,k)$



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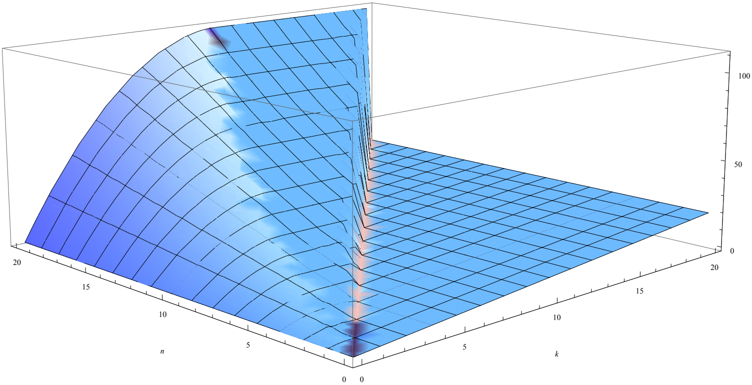

A conjecture refuted

E-S: a sequence of distinct numbers is *extremal* if it is of length $n^2 - 2n + 1$ and it has no monotonic n-subsequence.

The number of extremal sequences of a given length is always a square (RSK).

A conjecture refuted

A set partition is *extremal* if it is of weight $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor - 1$ and it has no free n-subpartition.

n	1	2	3	4	5	6
$f(n)$	1	2	4	6	9	12
$f(n)-1$	0	1	3	5	8	11
$t(n)$	1	1	1	4	9	121

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Sadly, $t(7) = 33^2 - 1$ and $t(8) = 298^2 - 16$ (Butler and Graham).

Future directions

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- Complexity: does a k-partition have a separated n-subpartition?
- Geometry

Geometry

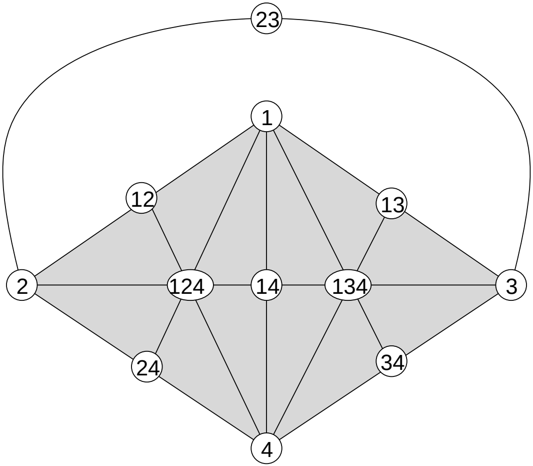

Figure : The free complex of 13/24.

Geometry

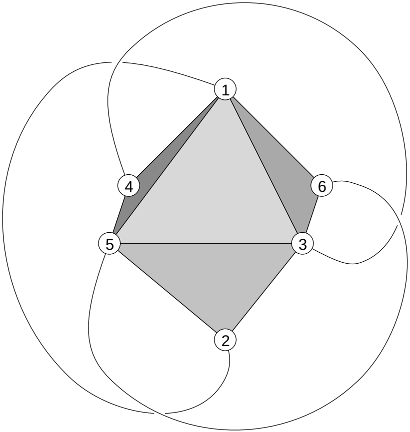

Figure : The monotonic complex of 563412.

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Gratitude

Thanks for your attention!