Towards a Solution to the Seymour Conjecture

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Outline

1. Background
   - The Seymour Conjecture
   - State of the Art
2. Proposed Approach
   - Definitions
   - Lemmas, Conjectures, and Theorems - Oh My!
   - Algorithm

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Background

The Seymour Conjecture

Definition

The k-th power of a graph G is the graph G^k with vertex set $V(G)$ and edge set

$$E(G^k) = \{xy \mid d_G(x,y) \leq k\}, \tag{1}$$

where $d_G(x,y)$ is the length of a shortest path in G between the vertices x and y.

Background
The Seymour Conjecture

- First proposed in 1974 by Paul Seymour, the conjecture is stated as follows:

Conjecture (Seymour)

If G is a graph of order n with minimum degree

$$\delta(G) \geq \frac{k}{k+1} n,$$

then there exists a Hamiltonian cycle H of G such that $H^k \subseteq G$.

Background

The Seymour Conjecture

- The Seymour conjecture generalizes Dirac's theorem ($k = 1$) and Posa's conjecture ($k = 2$).

Theorem (Dirac)

If G is a graph of order n and

$$\delta(G) \geq \frac{n}{2}, \tag{2}$$

then G is Hamiltonian.

Background
The Seymour Conjecture

- Seymour proposed the conjecture as a solution to a known-to-be-difficult theorem on coloring:

Theorem (Hajnal-Szemeredi)

If G is a graph with maximum degree

$$\Delta(G) \leq r, \tag{3}$$

then G has an equitable $(r+1)$*-coloring.*

- An equitable coloring is a proper coloring where the size of each color class differs by at most 1.

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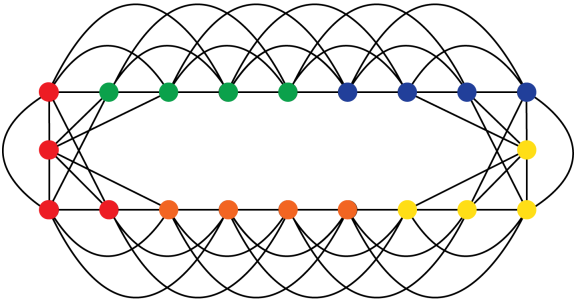
Background

The Seymour Conjecture

- The Seymour conjecture and Hajnal-Szemeredi theorem are complementary with respect to G.
- If one applies to G, then the other applies to G's complement $\overline{G}$.



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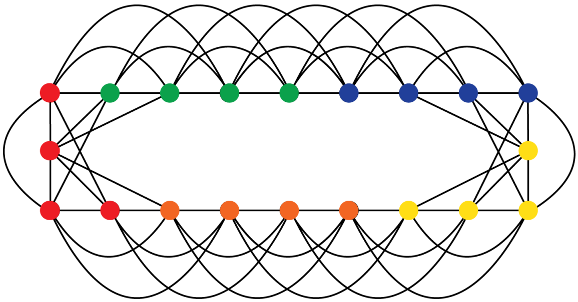

Background
The Seymour Conjecture

A complementary form of the Hajnal-Szemeredi theorem may be stated as follows:

Lemma (Complementary Hajnal-Szemeredi)

If G is a graph of order $n = sk$ and $\delta(G) \geq \frac{k-1}{k}n$, then G contains $\lfloor s \rfloor$ vertex-disjoint cliques of order k.

- A clique of order k, or k-clique, is a complete subgraph K_k.
- s is the number of colors $(r+1)$ from the Hajnal-Szemeredi theorem.
- There are s independent sets in $\overline{G}$.

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Background
State of the Art

- The best known results on the Seymour conjecture are vague approximations.

Theorem (Komlos-Sarkozy-Szemeredi)

For any $p > 0$ and positive integer k there is an $n_0 = n_0(p, k)$ such that, if $n \geq n_0$ and minimal degree

$$\delta(G) \geq \left(\frac{k}{k+1} + p\right) n, \tag{4}$$

then G contains the k-th power of a Hamiltonian cycle.

- Note: n_0 is on the order of 2^{100}.

Background
State of the Art

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Theorem (Faudree-Gould-Jacobson-Schelp)

For any $\epsilon > 0$ and positive integer k there is a $C(\epsilon, k)$ such that, if G is of order n with minimal degree

$$\delta(G) \geq \left(\frac{2k-1}{2k} + \epsilon\right) n + C(\epsilon, k), \qquad (4)$$

then G contains the k-th power of a Hamiltonian cycle.

- Note: A weaker bound than Seymour, and not even tight.

Background

Proposed Conjecture

- Goal is to prove following weaker version of the Seymour conjecture.
- Possibly extend results to full conjecture or other areas.

Conjecture

If G is a graph of order $n \geq 2k+1$ and

$$\delta(G) \geq \frac{2k-1}{2k}n, \tag{5}$$

then there exists a Hamiltonian cycle H of G such that $H^k \subseteq G$.

Proposed Approach

- The general idea is to employ the complementary Hajnal-Szemeredi lemma and analyze the connections between the cliques.
- However, the number of vertices and edges is problematic; a reduction to a simpler graph is desired.
- The level of detail is too fine; we wish to take a coarser look.

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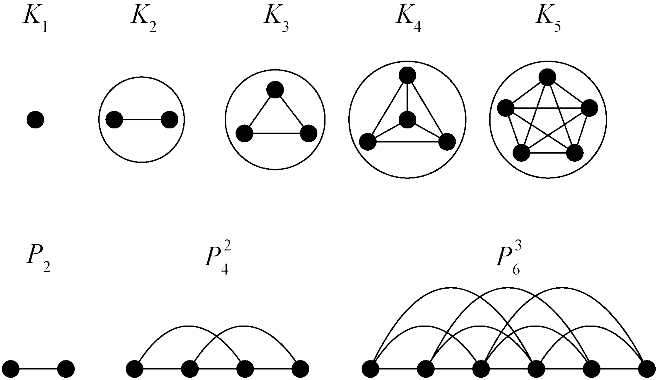
Definitions
Generalizing Vertex and Edge Definitions

Concepts including vertices and edges are generalized to larger structures.



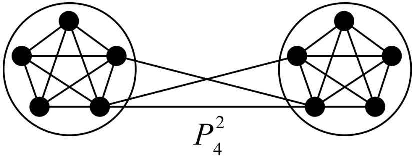
Definitions

Generalizing Vertex and Edge Definitions

Concepts including vertices and edges are generalized to larger structures.

Definition

Two vertex disjoint cliques C_1 and C_2 are defined to be (k, m)-path-adjacent if there exists a P_m^k subgraph P such that the first $\lceil m/2 \rceil$ vertices of P are elements of $V(C_1)$ and the last $\lfloor m/2 \rfloor$ vertices are elements of $V(C_2)$.



- The 5-cliques are $(2, 4)$-path-adjacent.

Definitions

Generalizing Vertex and Edge Definitions

Concepts including vertices and edges are generalized to larger structures.

Definition

A *k*-clique set of a graph G is a maximally-sized set $C_k(G)$ comprised of vertex disjoint cliques in G of order k. Note that a 2-clique set of G is a maximum matching.

- A k-clique set can be computed in polynomial time provided that $\delta(G) \geq \frac{k-1}{k} n$.

Clique Reductions
Definition

The clique reduction is the primary tool in (hopefully) deriving proof to the proposed conjecture.

Definition

A (k_1, k_2, m)-clique reduction of a graph G of order $n = sk_1$ is a graph $R(G)$ with vertex set $V(R(G)) = C_{k_1}(G)$, i.e. it has a vertex for each clique in $C_{k_1}(G)$, and edge set

$$E(R(G)) = \{xy | x, y \in C_{k_1}(G) \text{ and are } (k_2, m)\text{-path-adjacent}\} . \tag{6}$$

- The clique reduction and associated definitions provide a generalization of vertices and edges in a sense that preserves topology (don't quote me on this for rigor).

Clique Reductions
Notes

- The reduction is not necessarily unique; it relies upon the underlying k-clique set.
- Choice of parameters significantly alters reduced graph.
- The $(1, 1, 2)$-reduction of G is G.
- The $(k + 1, k, 2k)$-reduction is closely related to Seymour's conjecture..
- We focus on the $(2k, k, 2k)$-clique-reduction.

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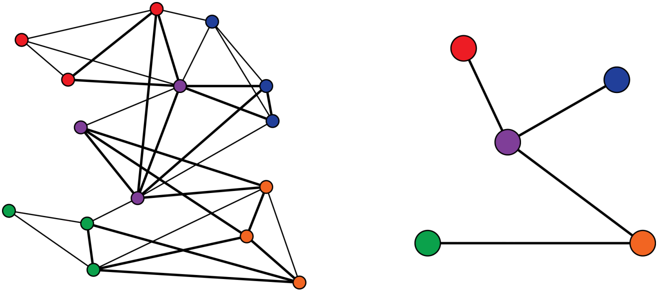
Clique Reductions
Example

A $(3, 2, 4)$-reduction of a Hamiltonian graph is shown below.



- Since this reduction is unique, we can deduce that there is no second power of a Hamiltonian path or cycle in the graph.

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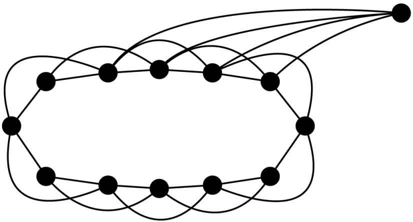
Incomplete Pigeonhole Principle

Lemma

If one has $n-2$ containers arranged in a line and at least $\frac{k}{k+1}n$ items that must be placed in distinct containers with $n \geq k+3$, then there exists an unbroken sequence of $k+1$ items.

- Mainly useful for proving that structures similar to the one below exist.



Main Results

General intuition: A Hamiltonian cycle in the reduced graph can be expanded into the k-th power of a cycle in the original graph.

Theorem

If a graph G has $n = 2sk$ vertices, with $s \geq 3$ and $k \geq 2$, and

$$\delta(G) \geq \frac{2k-1}{2k} n, \tag{7}$$

then the $(2k, k, 2k)$-clique reduction $R(G)$ is a Dirac graph.

- A Dirac graph is a graph with minimum degree $\delta(G) \geq n/2$.
- A Dirac approximation for this reduction can be computed in polynomial time.

Main Results

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Conjecture

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then there exists a Hamiltonian cycle of the $(2k, k, 2k)$-clique reduction that corresponds to a Hamiltonian cycle H of G such that $H^k \subseteq G$.

- Final piece for proving proposed conjecture.

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Potential Algorithm
Finding Hamiltonian Cycles

The proposed lemmas and conjectures hint at the possible existence of a polynomial time algorithm.

- Possibility hinges on existence of algorithm for Dirac graphs.
- Examination of the $(2, 1, 2)$-reduction is required – not guaranteed to be Dirac.

Lemma

If G is a Dirac graph with $n = 2s$ vertices, then there exists a Hamiltonian $(2, 1, 2)$-clique reduction.

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Potential Algorithm
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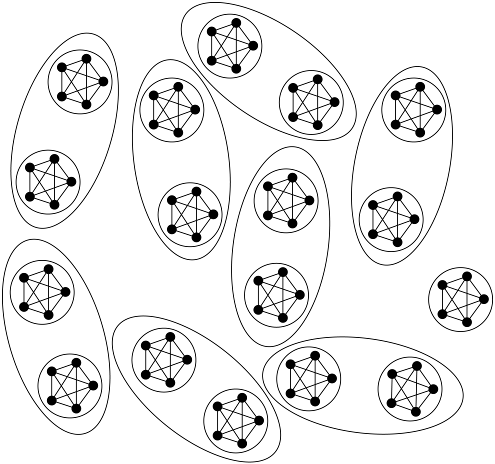
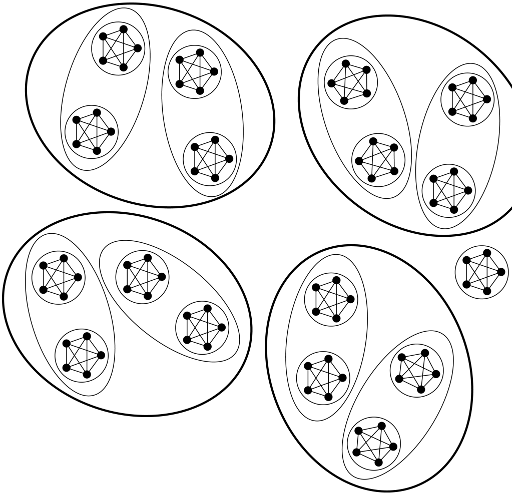
Suppose we start with a $(10, 5, 10)$-reduction.



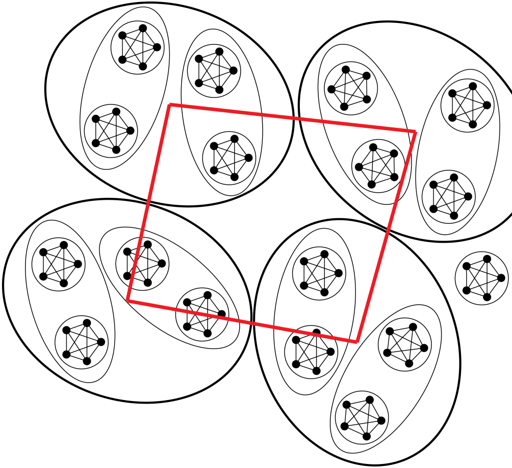
Potential Algorithm
Example

We take a $(2, 1, 2)$-reduction.



Potential Algorithm
Example

We find the Hamiltonian cycle in the smallest reduction . . .



Potential Algorithm
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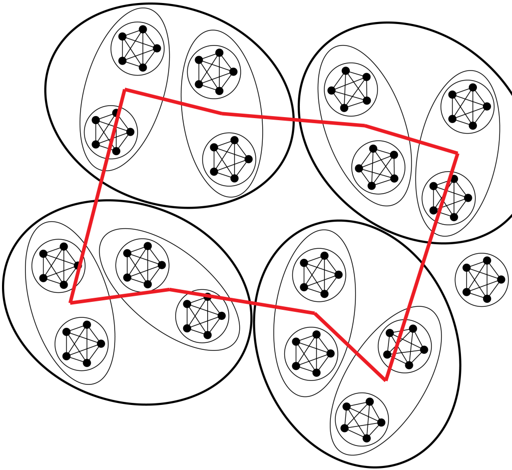
. . . and we expand the cycle.



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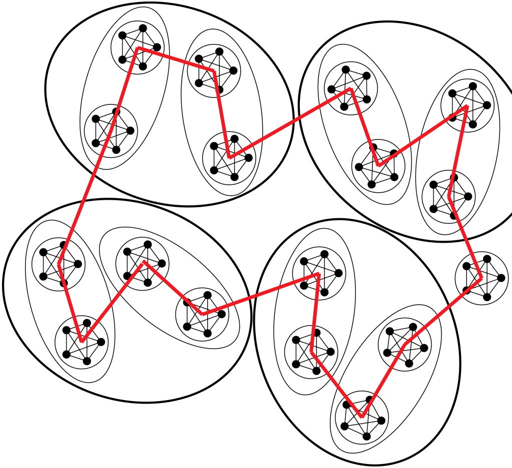
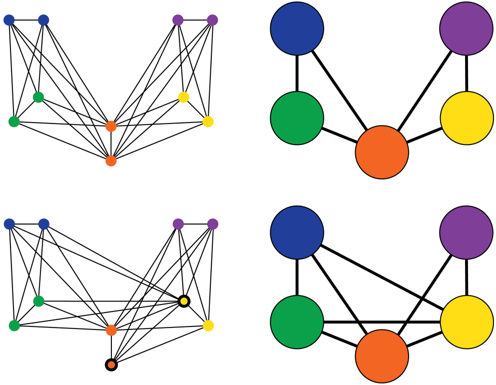


Conjecture

If G is a Dirac graph with $n = 2s$ vertices, then there exists an Ore $(2, 1, 2)$-clique reduction.

- Reduction not even guaranteed to be Hamiltonian, but simple transformations can correct.



Summary

- Future Outlook