# A combinatorial approach to some questions from affine algebraic geometry

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# Motivation from algebraic geometry

Let $\mathrm{GA_n}(\mathbb{C})$ denote the group of invertible polynomial maps $\mathbb{C}^n \to \mathbb{C}^n$.

- Shafarevich introduced the notion of an *infinite dimensional algebraic variety*, and showed that $\mathrm{GA_n}(\mathbb{C})$ has such a structure.
- A classical result of Jung is that $\mathrm{GA_2}(\mathbb{C})$ is an amalgamated free product of two subgroups; in particular, if $\theta \in \mathrm{GA_2}(\mathbb{C})$, then

$$\theta = \alpha_0 \tau_1 \alpha_1 \cdots \tau_r \alpha_r$$

  where $\tau_i$ are triangular, $\alpha_i$ are affine.
- The sequence $(\deg \tau_1, \cdots, \deg \tau_r)$ does not depend on the factorization and is called the *polydegree* of $\theta$.

# Motivation – the Polydegree Conjecture

Let $\mathcal{G}_d$ denote the subset of $\mathrm{GA}_2(\mathbb{C})$ of maps with polydegree $d = (d_1, \ldots, d_r)$.

### Question

*What conditions on the degree sequences $d$ and $e$ imply $\mathcal{G}_d \subset \overline{\mathcal{G}_e}$?*

### Conjecture (Polydegree Conjecture)

*Let $m, n \in \mathbb{N}$.*

$$\overline{\mathcal{G}_{(m+1,n+1)}} = \bigcup_{m' \leq m,\ n' \leq m} \mathcal{G}_{(m'+1,n'+1)} \cup \bigcup_{k \leq m+n+1} \mathcal{G}_{(k)}$$

Here, $\overline{\mathcal{G}_{(m+1,n+1)}}$ denotes the closure of $\mathcal{G}_{(m+1,n+1)}$ in the Zariski topology on $\mathrm{GA}_2(\mathbb{C})$.

# The Rigidity Conjecture

Furter showed that this is equivalent to the following:

**Conjecture (Rigidty Conjecture, R(m))**

*Let $m \in \mathbb{N}$, and let $a(T) \in \mathbb{C}[T]$ be a univariate polynomial of degree at most $m+1$ with the property that $a(T) \equiv T \pmod{T^2}$. If $m$ consecutive coefficients of the formal inverse $a^{-1}(T)$ vanish, then $a(T) = T$.*

Furter proved this for $m = 1, 2$.

# The Factorial Conjecture

Define a linear map $\mathcal{L} : \mathbb{C}^{[m]} \to \mathbb{C}$ by $\mathcal{L}(X_1^{\lambda_1} \cdots X_m^{\lambda_m}) = \lambda_1! \cdots \lambda_m!$.

Example

$\mathcal{L}(6X_1^2 - X_2^3 X_3^2) = 6(2!) - (3!)(2!) = 0$

Conjecture ((Weak) Factorial Conjecture)

*Let $f \in \mathbb{C}^{[m]}$ be nonzero. Then there exists $k \in \mathbb{N}$ such that $\mathcal{L}(f^k) \neq 0$.*

Example

$\mathcal{L}\left((6X_1^2 - X_2^3 X_3^2)^2\right) = 36(4!) - 12(2!)(3!)(2!) + (6!)(4!) > 0$

This question arose out of the work of van den Essen, Wright, and Zhao on several conjectures related to (and some of which are equivalent to) the Jacobian Conjecture.

# The Strong Factorial Conjecture

Define $f_m \in \mathbb{C}[Y_1, \ldots, Y_m][X_1, \ldots, X_m]$ by

$$f_m = X_1 \cdots X_m (X_1 Y_1 + \ldots + X_m Y_m).$$

Consider $\mathcal{L}$ as a $\mathbb{C}[Y_1, \ldots, Y_m]$-linear map $\mathcal{L} : \mathbb{C}^{[2m]} \to \mathbb{C}^{[m]}$

**Example**

$\mathcal{L}(f_m) = 2Y_1 + \cdots + 2Y_m$.

**Conjecture (Strong Factorial Conjecture, SFC(m))**

*Let $m \in \mathbb{N}$, let $\alpha \in \mathbb{C}^m \setminus \mathbf{0}$, and let $N(\alpha) < m$ be the number of nonzero components of $\alpha$. Then, for any $n \in \mathbb{N}$, $\mathcal{L}(f_m^i)(\alpha) \neq 0$ for some $n \leq i \leq n + N(\alpha) - 1$.*

**Theorem (Edo, van den Essen (2013))**

*SFC(m) is equivalent to the statement "R(m′) is true for all $m' \leq m$".*

# Towards a new conjecture

A few observations:

**Lemma**

*Let* $\lambda \in \mathbb{N}^m$. *Then for all* $\sigma \in \mathfrak{S}_m$, $\mathcal{L}(X^{\sigma(\lambda)}) = \mathcal{L}(X^{\lambda})$.

Can we view $f_m^n$ in terms of symmetric polynomials?

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Can we view $f_m^n$ in terms of symmetric polynomials? Yes.

$$
\begin{aligned}
(f_m)^n &= e_m(X_1, \ldots, X_m)^n m_{(1,\cdots,1)}(X_1 Y_1, \ldots, X_m Y_m)^n \\
&= e_m(X_1, \ldots, X_m)^n \sum_{\lambda \vdash n} \binom{n}{\lambda} m_\lambda(X_1 Y_1, \ldots, X_m Y_m)
\end{aligned}
$$

Here $\binom{n}{\lambda} = \binom{n}{\lambda_1 \lambda_2 \cdots \lambda_r}$ where $\lambda = (\lambda_1, \ldots, \lambda_r)$.

# Towards a new conjecture

## Lemma

*Let* $\lambda \in \mathbb{N}^m$, $n \in \mathbb{N}$. *Then*

$$\mathcal{L}(e_m(X_1, \ldots, X_m)^n m_\lambda(X_1 Y_1, \ldots, X_m Y_m)) = (\mathbf{n} + \lambda)! m_\lambda(Y)$$

(Here, $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^m$ and $(\mathbf{n} + \lambda)! = \prod_{i=1}^m (n + \lambda_i)!$)

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## Proof.

$$\begin{aligned}
\mathcal{L}(e_m^n m_\lambda) &= \mathcal{L}\left( X^{\mathbf{n}} \sum_{\nu \in \mathfrak{S}_m(\lambda)} X^\nu Y^\nu \right) \\
&= \sum_{\nu \in \mathfrak{S}_m(\lambda)} \mathcal{L}\left(X^{\mathbf{n}+\nu}\right) Y^\nu \\
&= (\mathbf{n} + \lambda)! \sum_{\nu \in \mathfrak{S}_m(\lambda)} Y^\nu \\
&= (\mathbf{n} + \lambda)! m_\lambda(Y)
\end{aligned}$$

# A new conjecture

We now define

$$g_{m,n}(Y) = \mathcal{L}(f_m^n) = \sum_{\lambda \vdash n} (\mathbf{n} + \lambda)! m_\lambda(Y)$$

## Conjecture (Strong Factorial Conjecture, SFC(m))

*Let $m \in \mathbb{N}$, let $\alpha \in \mathbb{C}^m \setminus \mathbf{0}$, and let $N(\alpha) < m$ be the number of nonzero components of $\alpha$. Then, for any $n \in \mathbb{N}$, $g_{m,k}(\alpha) \neq 0$ for some $n \leq k \leq n + N(\alpha) - 1$.*

## Conjecture (NC(m,n))

*Let $m, n \in \mathbb{N}$. Let $I = \sqrt{(g_{m,n}, \ldots, g_{m,n+m-1})}$. Then $I$ is a maximal ideal in the ring of symmetric polynomials.*

# Equivalence of the conjectures

**Theorem**

*Let $m \in \mathbb{N}$. $NC(m', n)$ is true for all $n \in \mathbb{N}$, $m' \leq m$ if and only if $SFC(m)$ is true.*

$\Rightarrow$: Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m$; assume $N(\alpha) = m$.

- $NC(m, n)$ implies $Y_1 \cdots Y_m \in I$
- For some some $r > 0$, we have

$$0 \neq (\alpha_1 \cdots \alpha_m)^r \in (g_{m,n}(\alpha), \ldots, g_{m,n+m-1}(\alpha))$$

- Thus $g_{m,n}(\alpha), \ldots, g_{m,n+m-1}(\alpha)$ cannot all be zero.

# Proof continued

$\Leftarrow$: Suppose $NC(m, n)$ does not hold:.

- $I \subset (e_1, \ldots e_m)$ is not a maximal ideal, so there is some other maximal ideal $(e_1 - \beta_1, \ldots, e_m - \beta_m)$ that contains $I$ as well.
- Choose $\alpha \in \mathbb{C}^m$ such that $e_i(\alpha) = \beta_i$; since $(\beta_1, \ldots, \beta_m) \neq \mathbf{0}$, $\alpha \neq \mathbf{0}$.
- Then $(g_{m,n}(\alpha), \ldots, g_{m,n+m-1}(\alpha)) \subset (0)$ (as an ideal of $\mathbb{C}$)
- This contradicts $SFC(m, n)$.

# Some more conjectures

It is trivial to check that the following implies NC(2,n):

### Conjecture

*For any* $n \in \mathbb{N}, \gcd(g_{2,n}, g_{2,n+1}) = 1$*.*

It is nontrivial but easy to show that the following implies the above conjecture:

### Conjecture

*If* $n$ *is even, then* $g_{2,n}$ *is irreducible. If* $n$ *is odd, then* $g_{2,n} = e_1 h_{2,n}$ *for some irreducible symmetric polynomial* $h_{2,n}$*.*

Mathematica calculations indicate the second conjecture is true for $n \leq 80$.

# One more conjecture

## Conjecture

*Let* $m, n \in \mathbb{N}$*. Then* $e_m^n \in (g_{m,n}, \ldots, g_{m,n+m-1})$

- A quick induction argument shows that this implies $NC(m, n)$
- Mathematica claims this conjecture holds for $m = 3, n \leq 30$.
- Open to attack via Gröbner basis methods?

# Questions

$$g_{m,n}(Y) = \sum_{\lambda \vdash n} (\mathbf{n} + \lambda)! m_\lambda(Y)$$

### Conjecture (NC(m,n))

*Let $m, n \in \mathbb{N}$. Let $I = \sqrt{(g_{m,n}, \dots, g_{m,n+m-1})}$. Then $I$ is a maximal ideal in the ring of symmetric polynomials.*

Some questions:

1. Is this question open?
2. Is it related to anything else in combinatorics?
3. Is it easy/hard?

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Thank you