Deer: Forbidden Minimal Digraphs

Stacey McAdams

Louisiana Tech University

Discrete Mathematics Workshop
Mississippi State University, Starkville, MS

Research with: Jinko Kanno

November 15-16, 2014

Outline

- Introduction
- Oriented Book Embeddings
- OBOP and Deer

Introduction

- A *graph*, $G = (V(G), E(G), f)$ is a triple consisting of
  - a vertex set $V(G)$,
  - an edge set $E(G)$,
  - a function f that associates each $e \in E(G)$ with a pair of vertices $\{v_1, v_2\} \in V(G)$; we say v_1 and v_2 are *adjacent*.
- A graph is *simple* if it contains no loops or multiple edges.
- All graphs in this talk are *simple* and *finite*

Introduction

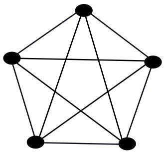
- A *graph*, $G = (V(G), E(G), f)$ is a triple consisting of
    - a vertex set $V(G)$,
    - an edge set $E(G)$,
    - a function f that associates each $e \in E(G)$ with a pair of vertices $\{v_1, v_2\} \in V(G)$; we say v_1 and v_2 are *adjacent*.
- A graph is *simple* if it contains no loops or multiple edges.
- All graphs in this talk are *simple* and *finite*

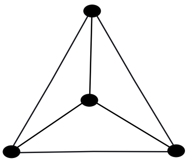
Introduction

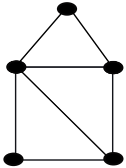
- A *graph*, $G = (V(G), E(G), f)$ is a triple consisting of
    - a vertex set $V(G)$,
    - an edge set $E(G)$,
    - a function f that associates each $e \in E(G)$ with a pair of vertices $\{v_1, v_2\} \in V(G)$; we say v_1 and v_2 are *adjacent*.
- A graph is *simple* if it contains no loops or multiple edges.
- All graphs in this talk are *simple* and *finite*

Introduction

- The *complete graph* on n vertices, denoted K_n, is a graph such that every pair of distinct vertices is adjacent.
- A graph is *planar* if it can be embedded in the plane such that no two edges cross. A graph is *outerplanar* if it is planar and every vertex lies on the boundary of the unbounded face.







Kuratowski's Theorem

Theorem (Kuratowksi, 1930)

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Corollary

A graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

Digraphs

- A *digraph*, or *directed graph*, $D = (V(D), A(D), \beta)$, is a triple consisting of
  - a vertex set $V(D)$,
  - an arc set $A(D)$, and
  - a function $\beta(a) = (v_i, v_j)$ that associates each arc with *exactly one* ordered pair of vertices.
- All digraphs in this talk are *finite* and have the property that if the directions were removed, the resulting graph would be *simple*.

Digraphs

- A *digraph*, or *directed graph*, $D = (V(D), A(D), \beta)$, is a triple consisting of
  - a vertex set $V(D)$,
  - an arc set $A(D)$, and
  - a function $\beta(a) = (v_i, v_j)$ that associates each arc with *exactly one* ordered pair of vertices.
- All digraphs in this talk are *finite* and have the property that if the directions were removed, the resulting graph would be *simple*.

Digraphs

- A *digraph*, or *directed graph*, $D = (V(D), A(D), \beta)$, is a triple consisting of
  - a vertex set $V(D)$,
  - an arc set $A(D)$, and
  - a function $\beta(a) = (v_i, v_j)$ that associates each arc with *exactly one* ordered pair of vertices.
- All digraphs in this talk are *finite* and have the property that if the directions were removed, the resulting graph would be *simple*.

Digraphs

- A *digraph*, or *directed graph*, $D = (V(D), A(D), \beta)$, is a triple consisting of
  - a vertex set $V(D)$,
  - an arc set $A(D)$, and
  - a function $\beta(a) = (v_i, v_j)$ that associates each arc with *exactly one* ordered pair of vertices.
- All digraphs in this talk are *finite* and have the property that if the directions were removed, the resulting graph would be *simple*.

Digraphs

- A *digraph*, or *directed graph*, $D = (V(D), A(D), \beta)$, is a triple consisting of
  - a vertex set $V(D)$,
  - an arc set $A(D)$, and
  - a function $\beta(a) = (v_i, v_j)$ that associates each arc with *exactly one* ordered pair of vertices.
- All digraphs in this talk are *finite* and have the property that if the directions were removed, the resulting graph would be *simple*.

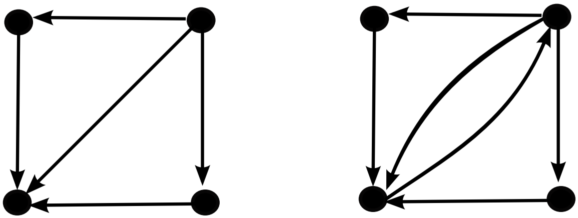


Figure : The left structure IS a digraph, the right IS NOT

Book Space

For $k > 0$, a *k-book* consists of

1. an oriented line L in $\mathbb{R}^3$, called the *spine*, and
2. k distinct half-planes, called *pages*, whose common boundary is L.

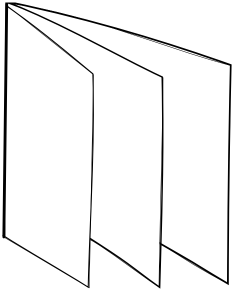


Figure : A 3-book

Book Space

For $k > 0$, a *k-book* consists of

1. an oriented line L in $\mathbb{R}^3$, called the *spine*, and
2. k distinct half-planes, called *pages*, whose common boundary is L.

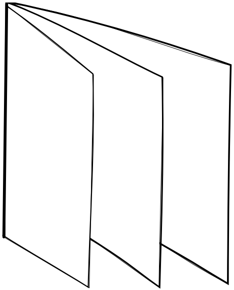


Figure : A 3-book

Book Space

For $k > 0$, a *k-book* consists of

1. an oriented line L in $\mathbb{R}^3$, called the *spine*, and
2. k distinct half-planes, called *pages*, whose common boundary is L.

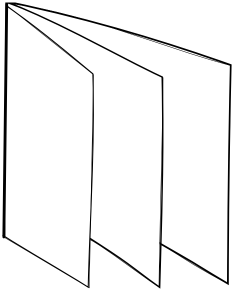


Figure : A 3-book

Book Space

A *k-book embedding* of a graph G is an embedding into a k-book such that:

1. the vertex set, $V(G)$, is a subset of L;
2. for each edge $e \in E(G)$:
   - e is a subset of the spine, or
   - e is a subset of at most one page, such that each page contains a planar subgraph of G,

The minimum number of pages required, so that no two edges cross on any page, is called the *book thickness*, denoted $bt(G)$.

Book Space

A *k-book embedding* of a graph G is an embedding into a k-book such that:

1. the vertex set, $V(G)$, is a subset of L;
2. for each edge $e \in E(G)$:
   - e is a subset of the spine, or
   - e is a subset of at most one page, such that each page contains a planar subgraph of G,

The minimum number of pages required, so that no two edges cross on any page, is called the *book thickness*, denoted $bt(G)$.

Book Space

A *k-book embedding* of a graph G is an embedding into a k-book such that:

1. the vertex set, $V(G)$, is a subset of L;
2. for each edge $e \in E(G)$:
   - e is a subset of the spine, or
   - e is a subset of at most one page, such that each page contains a planar subgraph of G,

The minimum number of pages required, so that no two edges cross on any page, is called the *book thickness*, denoted $bt(G)$.

Book Space

A *k-book embedding* of a graph G is an embedding into a k-book such that:

1. the vertex set, $V(G)$, is a subset of L;
2. for each edge $e \in E(G)$:
   - e is a subset of the spine, or
   - e is a subset of at most one page, such that each page contains a planar subgraph of G,

The minimum number of pages required, so that no two edges cross on any page, is called the *book thickness*, denoted $bt(G)$.

Book Space

A *k-book embedding* of a graph G is an embedding into a k-book such that:

1. the vertex set, $V(G)$, is a subset of L;
2. for each edge $e \in E(G)$:
    - e is a subset of the spine, or
    - e is a subset of at most one page, such that each page contains a planar subgraph of G,

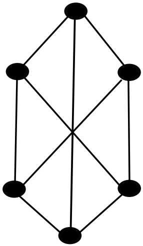
The minimum number of pages required, so that no two edges cross on any page, is called the *book thickness*, denoted $bt(G)$.

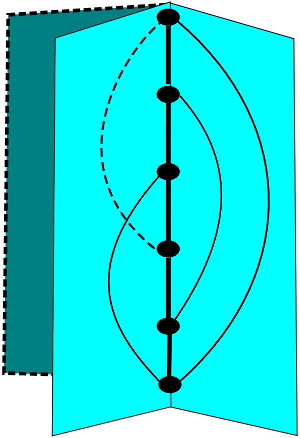
Book Space

A *k-book embedding* of a graph G is an embedding into a k-book such that:

1. the vertex set, $V(G)$, is a subset of L;
2. for each edge $e \in E(G)$:
   - e is a subset of the spine, or
   - e is a subset of at most one page, such that each page contains a planar subgraph of G,

The minimum number of pages required, so that no two edges cross on any page, is called the *book thickness*, denoted $bt(G)$.





Known Results on Book Embeddings

Theorem (Bernhart and Kainen, 1979)

Let G be connected, then the following hold.

- $bt(G) = 0$ *if and only if G is a path.*
- $bt(G) \leq 1$ *if and only if G is outerplanar.*
- $bt(G) \leq 2$ *if and only if G is a subgraph of an Hamiltonian planar graph.*

Theorem (Bernhart and Kainen, 1979)

For $4 \leq n$, $bt(K_n) = \lceil \frac{n}{2} \rceil$

Known Results on Book Embeddings

Theorem (Bernhart and Kainen, 1979)

Let G be connected, then the following hold.

- $bt(G) = 0$ *if and only if G is a path.*
- $bt(G) \leq 1$ *if and only if G is outerplanar.*
- $bt(G) \leq 2$ *if and only if G is a subgraph of an Hamiltonian planar graph.*

Theorem (Bernhart and Kainen, 1979)

For $4 \leq n$, $bt(K_n) = \lceil \frac{n}{2} \rceil$

Known Results on Book Embeddings

Theorem (Bernhart and Kainen, 1979)

Let G be connected, then the following hold.

- $bt(G) = 0$ *if and only if G is a path.*
- $bt(G) \leq 1$ *if and only if G is outerplanar.*
- $bt(G) \leq 2$ *if and only if G is a subgraph of an Hamiltonian planar graph.*

Theorem (Bernhart and Kainen, 1979)

For $4 \leq n$, $bt(K_n) = \lceil \frac{n}{2} \rceil$

Known Results on Book Embeddings

Theorem (Bernhart and Kainen, 1979)

Let G be connected, then the following hold.

- $bt(G) = 0$ *if and only if G is a path.*
- $bt(G) \leq 1$ *if and only if G is outerplanar.*
- $bt(G) \leq 2$ *if and only if G is a subgraph of an Hamiltonian planar graph.*

Theorem (Bernhart and Kainen, 1979)

For $4 \leq n$, $bt(K_n) = \lceil \frac{n}{2} \rceil$

Oriented Book Embeddings

An *oriented k-book embedding* of a digraph D is an embedding into a k-book such that :

1. the vertex set, $V(D)$, is a subset of L;
2. for each arc $a \in A(D)$:
   - a is a subset of the spine, or
   - a is a subset of at most one page, such that each page contains a planar subgraph of D,
3. all arcs within L share the same orientation as L;
4. Either all (nonspinal) arcs in a given page have the same orientation as L, or have the opposite orientation as L.

Oriented Book Embeddings

An *oriented k-book embedding* of a digraph D is an embedding into a k-book such that :

1. the vertex set, $V(D)$, is a subset of L;
2. for each arc $a \in A(D)$:
   - a is a subset of the spine, or
   - a is a subset of at most one page, such that each page contains a planar subgraph of D,
3. all arcs within L share the same orientation as L;
4. Either all (nonspinal) arcs in a given page have the same orientation as L, or have the opposite orientation as L.

Oriented Book Embeddings

An *oriented k-book embedding* of a digraph D is an embedding into a k-book such that :

1. the vertex set, $V(D)$, is a subset of L;
2. for each arc $a \in A(D)$:
    - a is a subset of the spine, or
    - a is a subset of at most one page, such that each page contains a planar subgraph of D,
3. all arcs within L share the same orientation as L;
4. Either all (nonspinal) arcs in a given page have the same orientation as L, or have the opposite orientation as L.

Oriented Book Embeddings

An *oriented k-book embedding* of a digraph D is an embedding into a k-book such that :

1. the vertex set, $V(D)$, is a subset of L;
2. for each arc $a \in A(D)$:
   - a is a subset of the spine, or
   - a is a subset of at most one page, such that each page contains a planar subgraph of D,
3. all arcs within L share the same orientation as L;
4. Either all (nonspinal) arcs in a given page have the same orientation as L, or have the opposite orientation as L.

Oriented Book Embeddings

An *oriented k-book embedding* of a digraph D is an embedding into a k-book such that :

1. the vertex set, $V(D)$, is a subset of L;
2. for each arc $a \in A(D)$:
   - a is a subset of the spine, or
   - a is a subset of at most one page, such that each page contains a planar subgraph of D,
3. all arcs within L share the same orientation as L;
4. Either all (nonspinal) arcs in a given page have the same orientation as L, or have the opposite orientation as L.

Oriented Book Embeddings

An *oriented k-book embedding* of a digraph D is an embedding into a k-book such that :

1. the vertex set, $V(D)$, is a subset of L;
2. for each arc $a \in A(D)$:
   - a is a subset of the spine, or
   - a is a subset of at most one page, such that each page contains a planar subgraph of D,
3. all arcs within L share the same orientation as L;
4. Either all (nonspinal) arcs in a given page have the same orientation as L, or have the opposite orientation as L.

Oriented Book Embeddings

An *oriented k-book embedding* of a digraph D is an embedding into a k-book such that :

1. the vertex set, $V(D)$, is a subset of L;
2. for each arc $a \in A(D)$:
   - a is a subset of the spine, or
   - a is a subset of at most one page, such that each page contains a planar subgraph of D,
3. all arcs within L share the same orientation as L;
4. Either all (nonspinal) arcs in a given page have the same orientation as L, or have the opposite orientation as L.

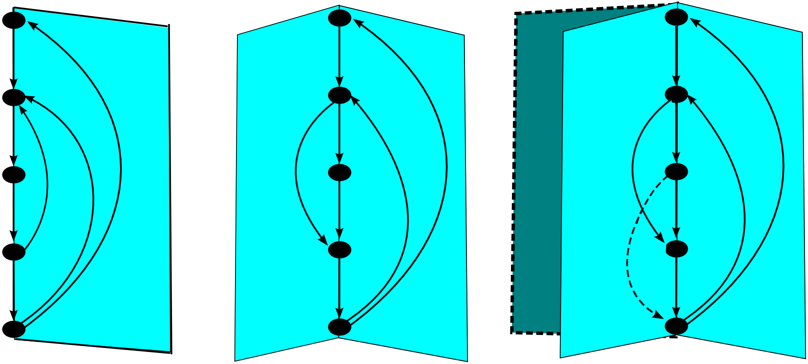


Figure : A 1-page, 2-page, and 3-page oriented book embedding.

Oriented Book Outerplanar

- A digraph is *oriented book outerplanar*, or OBOp, if it can be embedded onto a 1-book.
- **Our goal**: to characterize the class of OBOp digraphs.

Oriented Book Outerplanar

- A digraph is *oriented book outerplanar*, or OBOp, if it can be embedded onto a 1-book.
- **Our goal**: to characterize the class of OBOp digraphs.

Oriented Book Outerplanar (OBOp)

- To characterize OBOp digraphs, we first need the following three classes:
  - $\mathcal{R}$: A digraph $D \in \mathcal{R}$ if D is not OBOp and removing any arc in D results in an OBOp digraph.
  - $\mathcal{S}$: A digraph $D \in \mathcal{S}$ if D is not OBOp and switching the direction of any arc in D results in an OBOp digraph.
  - $\mathcal{M} = \mathcal{R} \cup \mathcal{S}$
- A digraph, D, is a *minimal forbidden digraph* for OBOp if $D \in \mathcal{M}$.

Oriented Book Outerplanar (OBOp)

- To characterize OBOp digraphs, we first need the following three classes:
  - $\mathcal{R}$: A digraph $D \in \mathcal{R}$ if D is not OBOp and removing any arc in D results in an OBOp digraph.
  - $\mathcal{S}$: A digraph $D \in \mathcal{S}$ if D is not OBOp and switching the direction of any arc in D results in an OBOp digraph.
  - $\mathcal{M} = \mathcal{R} \cup \mathcal{S}$
- A digraph, D, is a *minimal forbidden digraph* for OBOp if $D \in \mathcal{M}$.

Oriented Book Outerplanar (OBOp)

- To characterize OBOp digraphs, we first need the following three classes:
  - $\mathcal{R}$: A digraph $D \in \mathcal{R}$ if D is not OBOp and removing any arc in D results in an OBOp digraph.
  - $\mathcal{S}$: A digraph $D \in \mathcal{S}$ if D is not OBOp and switching the direction of any arc in D results in an OBOp digraph.
  - $\mathcal{M} = \mathcal{R} \cup \mathcal{S}$
- A digraph, D, is a *minimal forbidden digraph* for OBOp if $D \in \mathcal{M}$.

Oriented Book Outerplanar (OBOp)

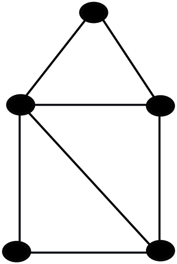
- To characterize OBOp digraphs, we first need the following three classes:
    - $\mathcal{R}$: A digraph $D \in \mathcal{R}$ if D is not OBOp and removing any arc in D results in an OBOp digraph.
    - $\mathcal{S}$: A digraph $D \in \mathcal{S}$ if D is not OBOp and switching the direction of any arc in D results in an OBOp digraph.
    - $\mathcal{M} = \mathcal{R} \cup \mathcal{S}$
- A digraph, D, is a *minimal forbidden digraph* for OBOp if $D \in \mathcal{M}$.

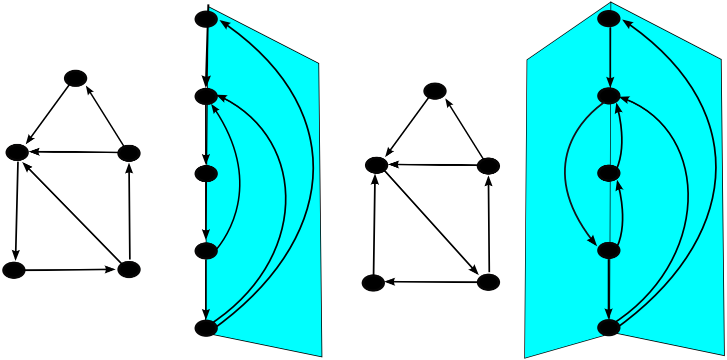
Oriented Book Outerplanar (OBOp)

- To characterize OBOp digraphs, we first need the following three classes:
    - $\mathcal{R}$: A digraph $D \in \mathcal{R}$ if D is not OBOp and removing any arc in D results in an OBOp digraph.
    - $\mathcal{S}$: A digraph $D \in \mathcal{S}$ if D is not OBOp and switching the direction of any arc in D results in an OBOp digraph.
    - $\mathcal{M} = \mathcal{R} \cup \mathcal{S}$
- A digraph, D, is a *minimal forbidden digraph* for OBOp if $D \in \mathcal{M}$.

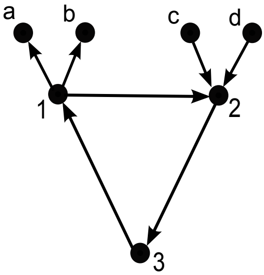
Characterizing OBOp

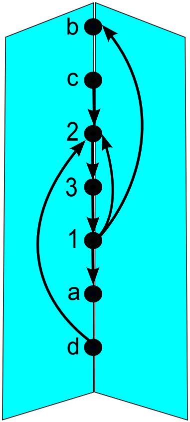
- By Bernhart and Kainen's Theorem, an outerplanar graph can be embedded onto a 1-page book; however, there exist outerplanar digraphs that are *not* OBOp.





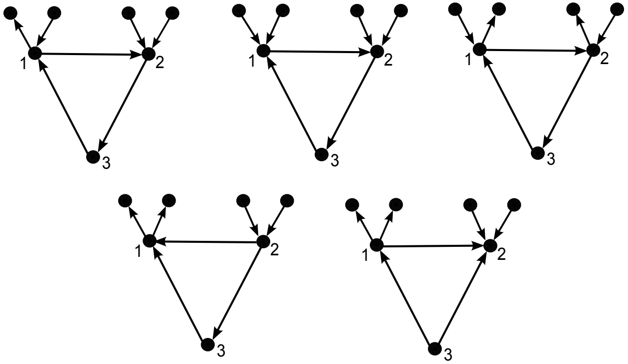
The Deer





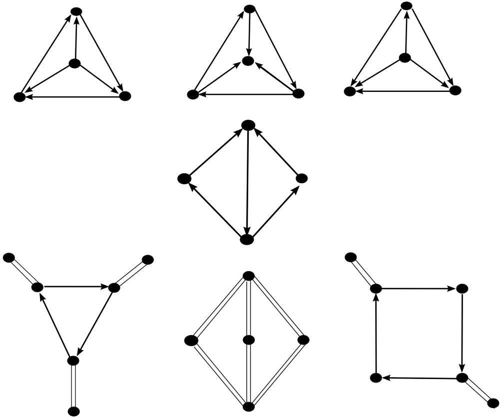
The Deer $\in \mathcal{S} \subseteq \mathcal{M}$.

OBOp



The above digraphs are OBOp.

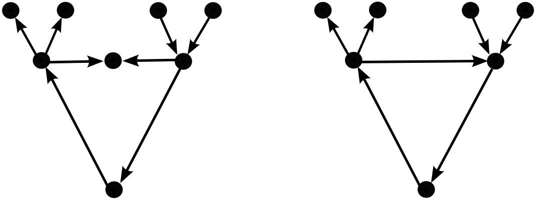
Other Digraphs in $\mathcal{M}$



- Double bar indicates arbitrary direction of the arc will still result in $D \in \mathcal{M}$.

Results

- For digraphs D_1, D_2, if $D_1 \in \mathcal{M}$ and $D_1 \leq D_2$, then D_2 is *not* OBOp.
- The class of OBOp digraphs is not minor closed. (1-page book embeddings for outerplanar graphs *is* minor closed)



Future Work

- Determine $\mathcal{M}$ completely.
- Extend results to oriented k-page book embeddings.

Thank you!!