Solvable regular covering projections of graphs

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Graph covers

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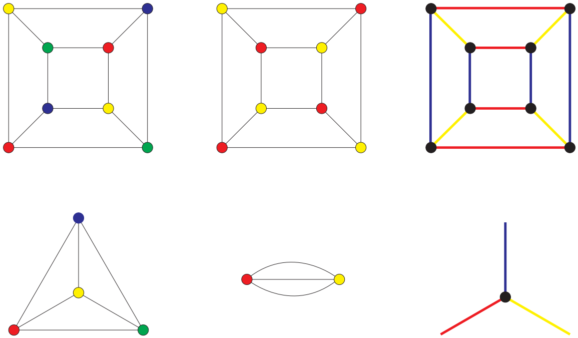
Regular coverings of connected graphs

a surjective mapping $p\colon \tilde{X} \to X$ s.t.
fibres $p^{-1}(v)$ = orbits of a semiregular subgroup $\mathrm{CT}_p \leq \mathrm{Aut}(\tilde{X})$



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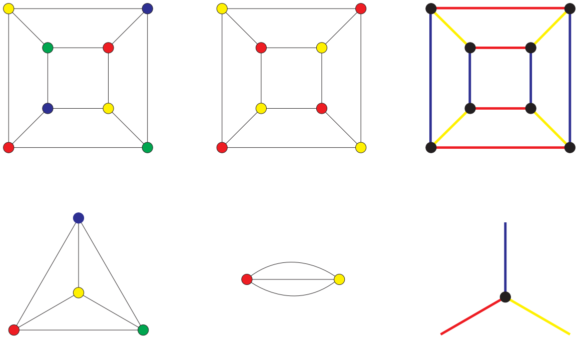

Construction/reconstruction

by a regular voltage function $\zeta\colon X \to \Gamma \cong CT_p$

Symmetries of covering graph vs. base graph

Lifting automorphisms along regular covering projections

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \\ p\downarrow & & \downarrow p \\ X & \xrightarrow[g]{} & X \end{array}$$

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all elements of $G \leq \mathrm{Aut}(X)$ **lift**

G-admissible regular cover

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Applications

classification of particular classes of graphs and maps on surfaces,
counting the number of graphs in certain families,
constructing infinite families or produce catalogues of graphs
with prescribed degree of symmetry up to a certain reasonable size

The structure of the lifted group

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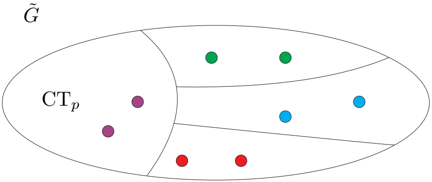

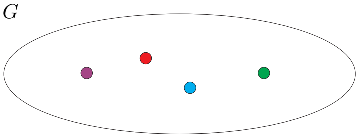

$\tilde{G}$ is a group extension of CT_p by G

$\mathrm{CT}_p \lhd \tilde{G}$ and $\tilde{G}/\mathrm{CT}_p \cong G$

Universal covering projection

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covering projection $p^*: \mathcal{T} \to X$

where $\mathcal{T}$ is a tree

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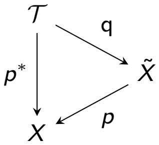
Universal property

for every $p\colon \tilde{X} \to X$ there exists a unique $q\colon \mathcal{T} \to \tilde{X}$ s.t.



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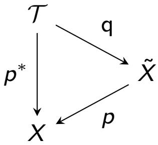

Additional properties

p^* is G-admissible for any $G \leq \mathrm{Aut}(X)$;
$\mathrm{CT}_{p^*} \cong \pi(X, u_0)$

$N \triangleleft \mathrm{CT}_{p^*} \longleftrightarrow$ regular covering projections

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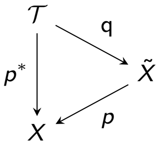
For a normal subgroup N of CT_{p^*}



$$\mathrm{CT}_p \cong \mathrm{CT}_{p^*}/N;$$
$$\mathrm{CT}_q \cong N \cong \pi(\tilde{X}, \tilde{u}_0)$$

Which $N \lhd \mathrm{CT}_{p^*}$ give rise to G-admissibility?

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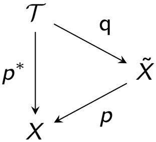

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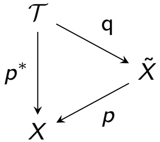

$$G^* = \text{the lifted group of } G \text{ along } p_{\zeta^*} (\mathrm{CT}_{p^*} \triangleleft G^* \text{ and } G^*/\mathrm{CT}_{p^*} \cong G)$$
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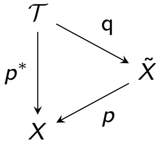

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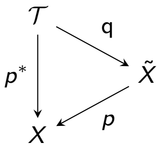

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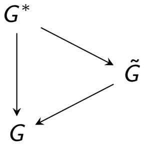
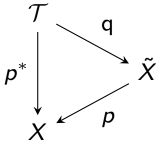
Then q is $\tilde{G}$ *− admissible*



Which $N \triangleleft \mathrm{CT}_{p^*}$ give rise to G-admissibility?



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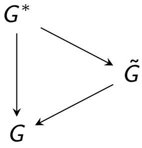

$N \triangleleft G^*$ **s.t.** $N \leq \mathrm{CT}_{p^*} \longleftrightarrow$ **G-admissible regular coverings**

Concrete computations?

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Finding a presentation of G^*

reconstruct p^* in terms of voltages,

use general recipe for constructing a presentation of G^* as an extension of CT_{p^*} by G,

find $G = \langle g_1, \ldots, g_n \mid r_1(g_1, \ldots, g_n), \ldots, r_m(g_1, \ldots, g_n) \rangle$,

use formula for evaluating lifts of automorphisms

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Up to a prescribed order n of the respective covering graphs

find all $N \lhd G^*$ contained in CT_{p^*} with CT_{p^*}/N solvable of order at most n

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find all $N \triangleleft G^*$ contained in CT_{p^*} with CT_{p^*}/N solvable of order at most n

The basic idea

in a solvable CT_{p^*}/N there exists a normal elementary abelian subgroup K/N;
if K is known, N can be found by considering $H \triangleleft G^*$
with $H \leq K$ and K/H elementary abelian

An algorithm

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Computing normal subgroups with solvable factor

Input: a finitely presented group G^*, a normal subgroup F of G^* given by words in the generators of G^* that generates F, an integer $n > 0$

Output: the set $\mathcal{N}$ of all normal subgroups N of G contained in H with F/N solvable of order at most n

Set $\mathcal{N} = \{F\}$ and set $\mathit{Processed} = \emptyset$;
while $\mathcal{N} \setminus \mathit{Processed} \neq \emptyset$ **do**
  Choose $K \in \mathcal{N} \setminus \mathit{Processed}$ and insert K in $\mathit{Processed}$;
  foreach prime p with $p|F : K| \leq n$ **do**
    Let $M = K/[K,K]K^p$ with $f : K \to M$ the natural epimorphisms;
    Turn M into $\mathbb{Z}_p[G^*/K]$-module;
    Find the set $\mathcal{S}$ of all maximal $\mathbb{Z}_p[G^*/K]$-submodules of M whose codimension d satisfies $p^d|F : K| \leq n$;
    foreach $S \in \mathcal{S}$ **do**
      Let $L = f^{-1}(S)$;
      if L is not equal to any of subgroups in $\mathcal{N}$ **then**
        Insert L into $\mathcal{N}$;
return $\mathcal{N}$;

First level

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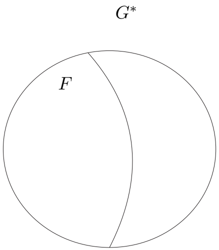

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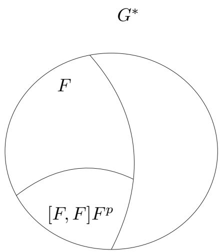

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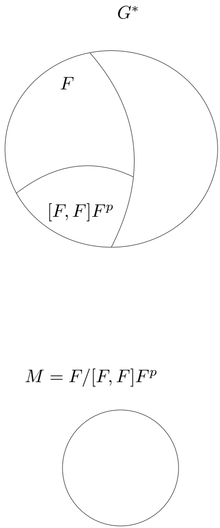

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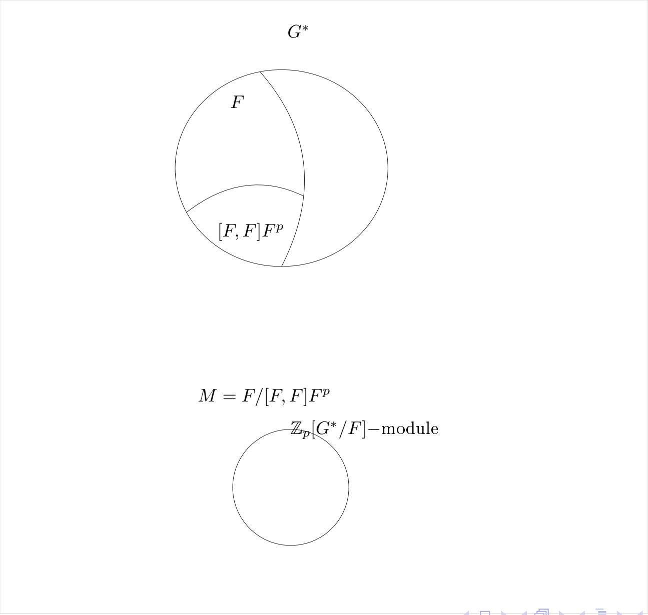

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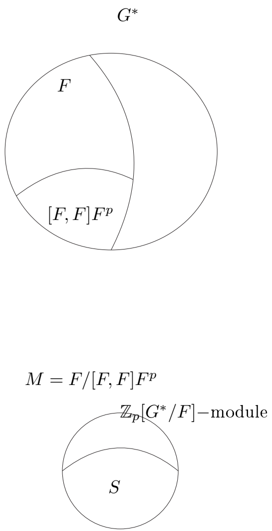

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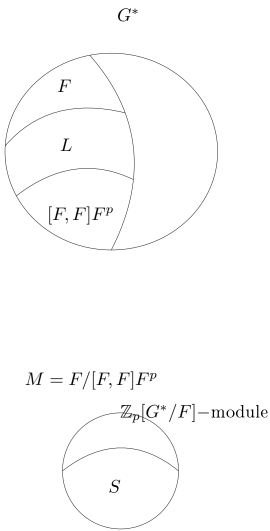

Thank you!