Padmakar-Ivan indices of k-trees

Shaohui Wang* and Bing Wei

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Outline

- Introduction

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- Some proofs

k-trees

Definition (Beineke and Pippert 1969)

The k-**tree**, denoted by T_n^k, for positive integers n, k with $n \geq k$, is defined recursively as follows:
The smallest k-tree is the k-clique K_k. If G is a k-tree with $n \geq k$ vertices and a new vertex v of degree k is added and joined to the vertices of a k-clique in G, then the obtained graph is a k-tree with $n+1$ vertices.

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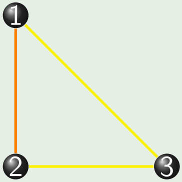

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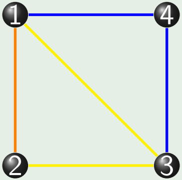

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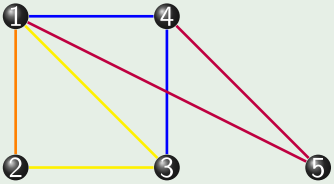

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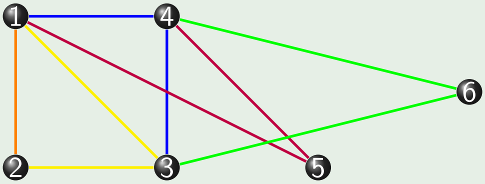

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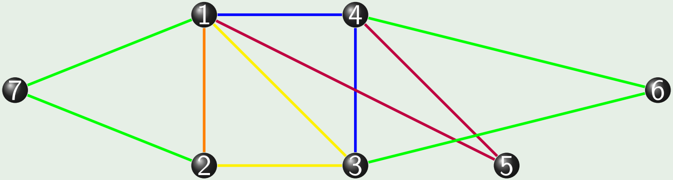

k-simplicial vertex

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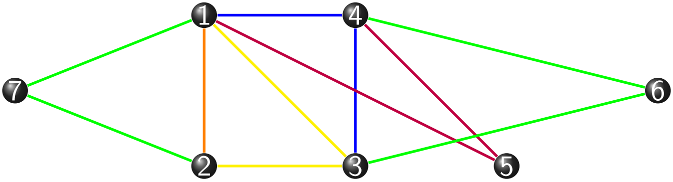
- In the following 2-tree, $5, 6, 7$ are 2-simplicial vertices.



k-simplicial vertex

- Let $S_1(T_n^k)$ be the set of all simplicial vertices of T_n^k, for $n \geq k+2$, and set $S_1(K_k) = \phi, S_1(K_{k+1}) = \{v\}$, where v is any vertex of K_{k+1}.

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- Let $G = G_0, G_i = G_{i-1} - v_i$, where v_i is a simplicial vertex of G_{i-1}, then $\{v_1, v_2 ... v_n\}$ is called **a simplicial elimination ordering** of the n-vertex graph G.

k-path and k-star

- The k-**path**, denoted by P_n^k, for positive integers n, k with $n \geq k$, is defined as follows:
  Starting with a k-clique $G[\{v_1, v_2...v_k\}]$. For $i \in [k+1, n]$, the vertex v_i is adjacent to vertices $\{v_{i-1}, v_{i-2}...v_{i-k}\}$ only.

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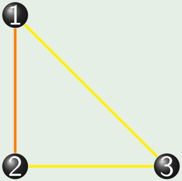

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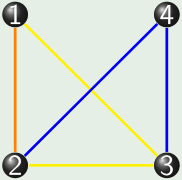

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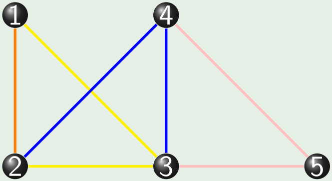

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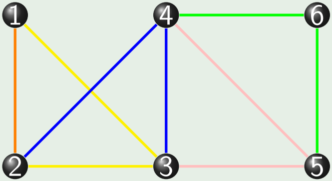

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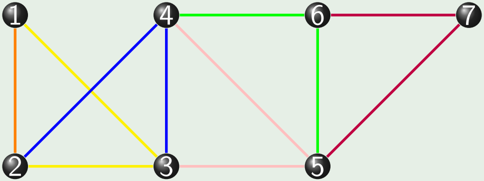

- The k-**star**, denoted by $S_{k,n-k}$, for positive integers n, k with $n \geq k$, is defined as follows:
  Starting with a k-clique $G[\{v_1, v_2 ... v_k\}]$ and an independent set S with $|S| = n - k$. For $i \in [k+1, n]$, the vertex v_i is adjacent to vertices $\{v_1, v_2 ... v_k\}$ only.

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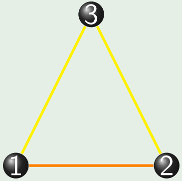

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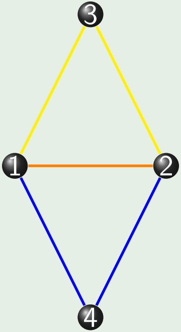

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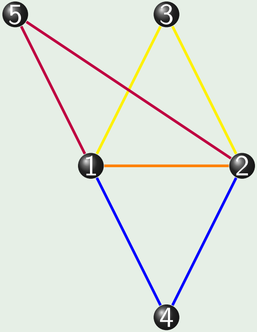

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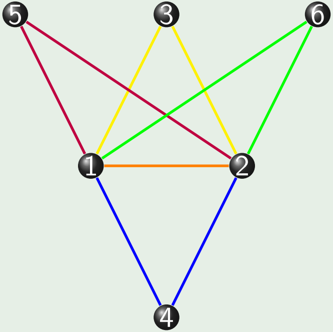

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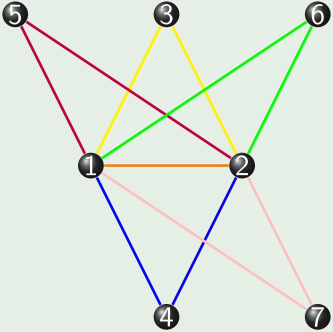

Zagreb Indices

Definition (Gutman and Trinajstić 1972)

The first and second **Zagreb indices** of the graph $G = (V, E)$ are defined as

$$M_1 = \sum_{v \in V(G)} d(v)^2; M_2 = \sum_{uv \in E(G)} d(u)d(v).$$

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Definition (Todeschini etc 2010; Wang, Wei 2014+)

The first(generalized) and second **Multiplicative Zagreb indices** of the graph $G = (V, E)$ are defined as

$$\prod_{1,c}(G) = \prod_{v \in V(G)} d(v)^c, c \geq 1; \prod_2(G) = \prod_{uv \in E(G)} d(u)d(v).$$

Theorem (Das and Gutman 2004)

Let T be any tree on n vertices, then

$$M_1(P_n) \leq M_1(T) \leq M_1(S_n),$$

$$M_2(P_n) \leq M_2(T) \leq M_2(S_n).$$

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Theorem (Estes and Wei 2012)

Let T_n^k be any k-tree on n vertices, then

$$M_1(P_n^k) \le M_1(T_n^k) \le M_1(S_{k,n-k}),$$

$$M_2(P_n^k) \le M_2(T_n^k) \le M_2(S_{k,n-k}).$$

Theorem (Gutman 2011)

Let $n \geq 5$ and T_n be any tree with n vertices, then

$$\prod_1(S_n) \leq \prod_1(T_n) \leq \prod_1(P_n),$$

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Theorem (Wang and Wei 2014+)

Let T_n^k be a k-tree on $n \geq k$ vertices, then

$$(1) \prod_{1,c}(S_{k,n-k}) \leq \prod_{1,c}(T_n^k) \leq \prod_{1,c}(P_n^k),$$

$$(2) \prod_2(P_n^k) \leq \prod_2(T_n^k) \leq \prod_2(S_{k,n-k}).$$

Definition (Winener 1947)

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For any $xy \in E(G)$, let $n_{xy}(x)$ be the number of vertices $w \in V(G)$ such that $d(x, w) < d(y, w)$.

Definition (Gutman 1994)

The **Szeged index** of the graph $G = (V, E)$ are defined as

$$Sz(G) = \sum_{xy \in E(G)} n_{xy}(x) n_{xy}(y).$$

Padmakar-Ivan Indices

Definition (Khadikar 2000)

The **Padmakar-Ivan index** of the graph $G = (V, E)$ is defined as

$$PI(G) = \sum_{xy \in E(G)} [n_{xy}(x) + n_{xy}(y)]$$

Padmakar-Ivan Indices

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$$PI(G) = \sum_{xy \in E(G)} [n_{xy}(x) + n_{xy}(y)]$$

- Let T be any tree on n vertices, respectively, then

$$PI(G) = (n-1)(n-2).$$

Main results

Theorem (1)

For any k-star S_n^k and k-path P_n^k with $n = kp + s$ vertices, where $2 \leq s \leq k+1$ and $p \geq 1$, then

$$(i) PI(S_n^k) = k(n-k)(n-k-1),$$
$$(ii) PI(P_n^k) = \frac{k(k+1)(p-1)(3kp+6s-2k-4)}{6} + \frac{(s-1)s(3k-s+2)}{3}.$$

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Theorem (2)

Let T_n^k be any k-tree on $n \geq k \geq 1$, then $PI(T_n^k) \leq PI(S_n^k)$.

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Let T_n^k be any k-tree on $n \geq k \geq 1$, then $PI(T_n^k) \leq PI(S_n^k)$.

Theorem (3)

There exists a k-tree T_n^k such that $PI(T_n^k) \leq PI(P_n^k)$, for any $k \geq 1$.

Let $d(u, v)$ be the distance between u and v in G, $d'(u, v)$ be the distance between u and v in G'. Build a fuction $f : \{xy \in E(G)\}$ to $\{1, 0\}$ as follows:

$$f(xy) = \begin{cases} 1 & \text{if} \quad d'(x, w) \neq d'(y, w), \\ 0 & \text{if} \quad d'(x, w) = d'(y, w). \end{cases}$$

Proof of Theorem 1.1

Let $V(G) = \{v_1, v_2, ..., v_n\}$, $G[v_1, ..., v_k]$ be a k-clique and $N(v_l) = \{v_1, v_2, ..., v_k\}$ for $l \geq k+1$. Just by the definition of k-star, we can get that for $i, j \in [1, k]$, $N[v_i] = N[v_j]$, then $PI(v_i v_j) = 0$; For $i \in [1, k]$ and $l \in [k+1, n]$, $|N[v_i] - N[v_l]| = n - k - 1$, then $PI(v_i v_l) = n - k - 1$. Thus, we can get $PI(G) = \sum_{i,j \in [1,k]} PI(v_i v_j) + \sum_{i \in [1,k], l \in [k+1,n]} PI(v_i v_l) = k(n-k)(n-k-1)$.

Proof of Theorem 2

- Let xy be any edge of a k-tree G with at least $n \geq k+1$ vertices, then $PI(xy) \leq n-k-1$.

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- By induction.

- T_n^{k*}: Let $v_1, v_2, ..., v_{n-k-1}$ be the simplicial ordering of P_{n-1}^{k-1} and T_n^{k*} be a k-tree with n vertices such that $T_n^{k*} = P_{n-1}^{k-1} \bigoplus \{v_n\}$, that is, $V(T_n^{k*}) = \{v_1, v_2, ..., v_n\}$ and $E(T_n^{k*}) = E(P_{n-1}^{k-1}) \cup \{v_1 v_n, v_2 v_n, ..., v_{n-1} v_n\}$.

Some Remarks

- T_n^{k*}: Let $v_1, v_2, ..., v_{n-k-1}$ be the simplicial ordering of P_{n-1}^{k-1} and T_n^{k*} be a k-tree with n vertices such that $T_n^{k*} = P_{n-1}^{k-1} \bigoplus \{v_n\}$, that is, $V(T_n^{k*}) = \{v_1, v_2, ..., v_n\}$ and $E(T_n^{k*}) = E(P_{n-1}^{k-1}) \cup \{v_1 v_n, v_2 v_n, ..., v_{n-1} v_n\}$.
- The 2-tree T_6^2 such that $V(T_6^2) = \{v_1, v_2, ..., v_6\}$ and $E(T_6^2) =$ $\{v_1 v_2, v_2 v_3, v_3 v_1\} \cup \{v_4 v_1, v_4 v_2, v_5 v_2, v_5 v_3, v_6 v_3, v_6 v_1\}$, then $PI(T_6^2) = 18$. $PI(T_6^{2*}) = 18$, then $PI(T_n^{k*})$ is still not less than any PI-value of other type of k-trees.

Thank you