Dichromatic Number and Fractional Chromatic Number of a Graph

Hehui Wu

University of Mississippi

Starkville MS, Nov 16th 2014

Joint work with Bojan Mohar

A graph is k-colorable if it exists a partition of vertices into independent set.

A chromatic number of a graph G is the minimum order of partitions of the graph into k independent sets, denoted by $\chi(G)$.

A graph is k-colorable if it exists a partition of vertices into independent set.

A chromatic number of a graph G is the minimum order of partitions of the graph into k independent sets, denoted by $\chi(G)$.

A digraph is k-colorable if it exists a partition of vertices into k acyclic sets.

A chromatic number of a digraph D is the minimum order of partitions of the vertices into acyclic set, denoted by $\chi(D)$. (Neumann-Lara)

A digraph is k-colorable if it exists a partition of vertices into k acyclic sets.

A chromatic number of a digraph D is the minimum order of partitions of the vertices into acyclic set, denoted by $\chi(D)$. (Neumann-Lara)

Definition of Chromatic Number of Undirected Graphs and Digraphs

A digraph is k-colorable if it exists a partition of vertices into k acyclic sets.

A chromatic number of a digraph D is the minimum order of partitions of the vertices into acyclic set, denoted by $\chi(D)$. (Neumann-Lara)

An ordering π of a set A of order n is an one-to-one mapping from A to $[n]$.

A backward graph G of a digraph D in the ordering π is the graph with the same vertex set as D, and $uv \in E(G)$ if and only if uv is an arc of D and $\pi(u) > \pi(v)$.

A digraph is k-colorable if it exists a partition of vertices into k acyclic sets.

A chromatic number of a digraph D is the minimum order of partitions of the vertices into acyclic set, denoted by $\chi(D)$. (Neumann-Lara)

An ordering π of a set A of order n is an one-to-one mapping from A to $[n]$.

A backward graph G of a digraph D in the ordering π is the graph with the same vertex set as D, and $uv \in E(G)$ if and only if uv is an arc of D and $\pi(u) > \pi(v)$.

Observation

A digraph is acyclic if and only if the backward graph for some ordering is an independent set. An digraph is k-colorable if and only if it is k-colorable for the backward graph in some ordering.

$\mathcal{D}(G)$: digraphs obtains from an orientation of the edges of G.
The dichromatic number of an undirected graph G, denoted by dichromatic number $\overrightarrow{\chi}(G)$, is $\max\{\chi(D) : D \in \mathcal{D}(G)\}$ (Erdős-Neumann Lara, 1978)

$\mathcal{D}(G)$: digraphs obtains from an orientation of the edges of G.
The dichromatic number of an undirected graph G, denoted by dichromatic number $\overrightarrow{\chi}(G)$, is $\max\{\chi(D) : D \in \mathcal{D}(G)\}$ (Erdős-Neumann Lara, 1978)
It is known that

$$\overrightarrow{\chi}(K_n) = \begin{cases} 2, & n = 3, 4, 5, 6 \\ 3, & n = 7, 8, 9, 10 \\ 4, & n = 11 \cdots \end{cases}$$

$\mathcal{D}(G)$: digraphs obtains from an orientation of the edges of G.
The dichromatic number of an undirected graph G, denoted by dichromatic number $\overrightarrow{\chi}(G)$, is $\max\{\chi(D) : D \in \mathcal{D}(G)\}$ (Erdős-Neumann Lara, 1978)
It is known that

$$\overrightarrow{\chi}(K_n) = \begin{cases} 2, & n = 3, 4, 5, 6 \\ 3, & n = 7, 8, 9, 10 \\ 4, & n = 11 \cdots \end{cases}$$

"It is surprisingly difficult to $\overrightarrow{\chi}(G)$ even for the simplest graphs."—-Erdős

Theorem (Erdos-Neumann Lara, 1978)

There exists positive constants c_1 and c_2, such that $c_1 \frac{n}{\log n} \leq \overrightarrow{\chi}(K_n) \leq c_2 \frac{n}{\log n}$.

$\mathcal{D}(G)$: digraphs obtains from an orientation of the edges of G.
The dichromatic number of an undirected graph G, denoted by dichromatic number $\overrightarrow{\chi}(G)$, is $\max\{\chi(D) : D \in \mathcal{D}(G)\}$ (Erdős-Neumann Lara, 1978)
It is known that

$$\overrightarrow{\chi}(K_n) = \begin{cases} 2, & n = 3, 4, 5, 6 \\ 3, & n = 7, 8, 9, 10 \\ 4, & n = 11 \cdots \end{cases}$$

"It is surprisingly difficult to $\overrightarrow{\chi}(G)$ even for the simplest graphs."—-Erdős

Theorem (Erdos-Neumann Lara, 1978)

There exists positive constants c_1 and c_2, such that $c_1 \frac{n}{\log n} \leq \overrightarrow{\chi}(K_n) \leq c_2 \frac{n}{\log n}$.

Theorem (Erdős-Gimbel-Kratsch, 1991)

There exists positive constants c_1, c_2, such that

$$c_1 k^2 \log^2 k \leq \min\{e(G) : \overrightarrow{\chi}(G) = k\} \leq c_2 k^2 \log^2 k.$$

Problems and results in number theory and graph theory

Paul Erdős

First I discuss some problems on the iteration of number theoretic functions.

I. There are many attractive and amusing problems in this subject but almost no definitive results. One of the best known conjectures in this subject is an old conjecture of Catalan. Put

$$\sigma_1(n) = \sigma(n) - n, \qquad \sigma(n) = \sum_{d|n} d, \qquad \sigma_k(n) = \sigma_1(\sigma_{k-1}(n)).$$

Catalan conjectured that the sequence $\{\sigma_k(n)\}$, $k = 1,2,\ldots$ is bounded for every n, in other words it either leads to 1 or to a cycle. The Lehmers, Guy, Selfridge and Wunderlich have a great deal of numerical evidence about this conjecture; Guy and Selfridge have good heuristic evidence that the conjecture is probably false, in fact probably false for almost all even numbers n. The computations of the Lehmers seem to indicate that the conjecture is probably false for $n = 276$, but so far no proof is in sight and I do not expect any breakthrough in the near (or distant) future. I proved that for fixed k and almost all n

(1) $$\sigma_k(n) = (1 + o(1))n\left(\frac{\sigma_1(n)}{n}\right)^k.$$

It is clear that (1) gives no help at all in deciding Catalan's conjecture. I do not see at all how to estimate $\sigma_n(n)$ or even $\sigma_{[\log n]}(n)$.

PROC. NINTH MANITOBA CONFERENCE ON
NUMERICAL MATH. AND COMPUTING, 1979, pp. 3-21.

III. Finally I report on some results V. Neumann-Lara and I found over the last two years; detailed proofs will be published elsewhere. First some notations. $G(n;e)$ will denote a graph of n vertices and e edges. $G(n)$ a graph of n vertices and G_e a graph of e edges. Let G be a directed graph. Neumann-Lara defines $d_k(G)$, the dichromatic number of G, as the smallest integer so that the vertex set of G can be decomposed into $d_k(G)$ disjoint subsets none of which span a directed circuit. He is preparing a paper on his function $d_k(G)$. When I first visited Mexico City two years ago we started to investigate a modification of this function. Let G be (an undirected) graph. The dichromatic number $d_k(G)$ is the smallest integer so that for any orientation of the edges of G one can always divide the vertex set into $d_k(G)$ or fewer disjoint sets, none of which span a directed circuit of G (in the given orientation).

It is surprisingly difficult to determine $d_k(G)$ even for the simplest graphs. We proved ($k(n)$ is the complete graph of n vertices)

(1) $$c_1 n/\log n < d_k(k(n)) < c_2 n/\log n.$$

Probably there is no simple explicit formula $d_k(k(n))$. We could not even prove that

(2) $$d_k(k(n))\log n \cdot n^{-1} \to c.$$

Let $f(n)$ be the smallest integer for which there is a $G_{f(n)}$ of dichromatic number n. Perhaps then $G_{f(n)}$ must be a complete graph. This is easy to prove for the ordinary chromatic number but we could not prove it for the dichromatic number, and in fact we could not even prove that $f(n)/n^2$ tends to infinity - we have no doubt that this is true.

Theorem (Erdos-Neumann Lara, 1978)

There exists positive constants c_1 and c_2, such that $c_1 \frac{n}{\log n} \leq \overrightarrow{\chi}(K_n) \leq c_2 \frac{n}{\log n}$.

A tournament is an orientation of a complete graph.

Theorem

Almost all tournaments of order k have chromatic number at least $\frac{1}{2}\left(\frac{n}{\log n+1}\right)$.

Theorem (Erdos-Neumann Lara, 1978)

There exists positive constants c_1 and c_2, such that $c_1 \frac{n}{\log n} \leq \overrightarrow{\chi}(K_n) \leq c_2 \frac{n}{\log n}$.

A tournament is an orientation of a complete graph.

Theorem

Almost all tournaments of order k have chromatic number at least $\frac{1}{2}\left(\frac{n}{\log n+1}\right)$.

Proof.

Randomly pick an tournament D on $[n]$. For any $2(\log n + 1)$ vertices A in $[n]$.

$$Pr(A \text{ is acyclic}) = \frac{(\log n + 1)!}{2^{\binom{2\log n+2}{2}}}.$$

There are $\binom{n}{\log n+1}$ such sets,

$Pr(\text{some set of size } \log n + 1 \text{ is acyclic}) \leq \binom{n}{2\log n+2} \frac{(2\log n+2)!}{2^{\binom{2\log n+2}{2}}} < 1/n.$

Hence for almost all tournaments, there is no acyclic set have size more than $2(\log n + 1)$, hence the chromatic number will be at least $\frac{n}{2\log n+2}$. □

Conjecture (McDiarmid-Mohar, 2002)

Every graph G with maximum degree Δ has $\vec{\chi}(G) \leq O(\frac{\Delta}{\log \Delta})$.

Conjecture (Erdos-Neumann Lara, 1978)

For any positive integer k, there exists $r(k)$, such that if $\vec{\chi}(G) \leq k$, then $\chi(G) \leq r(k)$.

Conjecture (Berger-Choromanski-Chudnovsky-Fox-Loebl-Scott-Seymour-Thomassé, 2013)

For any k, there exists $f(k)$, such that for any tournament, if the out-neighborhood of any vertex has chromatic number at most k, then the tournament has chromatic number at most $f(k)$.

Results

Theorem (Harutyunyan-Mohar)

For any big enough integers g and Δ, there exists digraph D of girth at least g, with $\Delta(G) \leq \Delta$, and $\chi(D) \geq a\Delta/\log\Delta$ for some positive constant a.

Theorem (Harutyunyan-Mohar)

For every k, there exists $\epsilon > 0$ such that for every sufficiently large integer n, there exists a digraph D of order n with $\chi(G) \geq k$ and $\chi(D[S]) \leq 2$ for every $S \subset V(D)$ with $|S| \leq \epsilon n$.

Theorem (Harutyunyan-Mohar)

For any big enough integers g and Δ, there exists digraph D of girth at least g, with $\Delta(G) \leq \Delta$, and $\chi(D) \geq a\Delta/\log\Delta$ for some positive constant a.

Theorem (Harutyunyan-Mohar)

For every k, there exists $\epsilon > 0$ such that for every sufficiently large integer n, there exists a digraph D of order n with $\chi(G) \geq k$ and $\chi(D[S]) \leq 2$ for every $S \subset V(D)$ with $|S| \leq \epsilon n$.

Theorem (Mohar-W., 2014+)

If $\chi_f(G) \geq k$, then $\overrightarrow{\chi}(G) \geq \overrightarrow{\chi}_f(G) \geq \frac{t}{18\log t}$.

Let $\mathcal{I}$ be the collection of independent sets.
The fractional chromatic number of G, denoted by $\chi_f(G)$ is the optimal solutions for the following dual linear programs:

Minimize	$\sum_{I\in\mathcal{I}(G)} x_I$		Maximize	$\max \sum_{v\in V} y_v$	
such that			such that		
	$\forall\, v \in V$	$\sum_{I\in\mathcal{I}(G,x)} x_I \geq 1$		$\forall I \in \mathcal{I}(G)$	$\sum_{v\in I} y_v \leq 1$
	$\forall\, I \in \mathcal{I}(G)$	$0 \leq x_I \leq 1$		$\forall\, v \in V$	$0 \leq y_v \leq 1$

Theorem

$\chi_f(G) \geq x$ if and only if there exists a weight functions w on vertices, such that the total weight is x and each independent set has weight at most 1.

Theorem (Mohar-W., 2014+)

If $\chi_f(G) \geq k$, *then* $\vec{\chi}(G) \geq \vec{\chi}_f(G) \geq \frac{t}{18 \log t}$.

Theorem (Mohar-W., 2014+)

If $\chi_f(G) \geq k$, *then* $\vec{\chi}(G) \geq \vec{\chi}_f(G) \geq \frac{t}{18 \log t}$.

Kneser graph $G = KG(n,k) : V(G) = \binom{[n]}{k}, E(G) = \{AB : A \cap B = \emptyset\}$.

Theorem

$\chi(KG(n,k)) = n - 2k + 2, \chi_f(KG(n,k)) = \frac{n}{k}$.

Theorem (Mohar-W., 2014+)

If G *is a Kneser Graph* $KG(n,k)$, *then* $\vec{\chi}(G) \geq \left\lfloor \frac{\chi(G)}{8 \log(\chi_f(G))} \right\rfloor$.

Corollary

For any $\epsilon > 0$ *and arbitrary large* k, *there exists a graphs* G *with* $\vec{\chi}(G) \geq k$ *and* $\chi_f(G) \leq 2 + \epsilon$.

Theorem (Mohar-W., 2014+)

If $\chi_f(G) \geq k$, *then* $\overrightarrow{\chi}(G) \geq \overrightarrow{\chi}_f(G) \geq \frac{t}{18 \log t}$.

Theorem (Mohar-W., 2014+)

If $\chi_f(G) \geq k$, *then* $\vec{\chi}(G) \geq \vec{\chi}_f(G) \geq \frac{t}{18 \log t}$.

Let $\mathcal{AD}(G)$ be the family of acyclic orientations of G.

Lemma

$|\mathcal{AD}(G)| \leq \prod_{v \in V(G)} (d_G(v) + 1)$.

Proof.

Let $f : \mathcal{AD}(G) \longmapsto \mathbb{Z}^n$, such that $f(D) = (d_D^+(v_1), \cdots, d_D^+(v_n))$.
For $D_1, D_2 \in \mathcal{AD}(G)$, $f(D_1) = f(D_2)$ if and only if $D_1 = D_2$.
Since $0 \leq d_D^+(v_i) \leq d_G(v_i)$, there are at most $\prod_{v \in V(G)} (d_G(v) + 1)$ possible value for $f(D)$.
Therefore $|\mathcal{AD}(G)| \leq \prod_{v \in V(G)} (d_G(v) + 1)$. □

Theorem (Mohar-W., 2014+)

If $\chi_f(G) \geq k$, then $\vec{\chi}(G) \geq \vec{\chi}_f(G) \geq \frac{t}{18 \log t}$.

Let $\mathcal{AD}(G)$ be the family of acyclic orientations of G.

Lemma

$|\mathcal{AD}(G)| \leq \prod_{v \in V(G)} (d_G(v) + 1)$.

The average degree $\overline{d}(G) = \frac{\sum_v d_G(v)}{n} = 2e(G)/n$.

Corollary

If $\overline{d}(G) \geq 17$, then $Pr_{D \in \mathcal{D}(G)}(D \text{ is acyclic}) < 2^{-e(G)/2}$.

Proof outline

$$X_f(G) = t \Longrightarrow \quad \begin{array}{l} \exists w : V(G) \mapsto R_0^+, \text{ such that:} \\ (a) w(V) = t, \\ (b) \text{ for any independent set } I, w(I) \leq 1. \end{array}$$

Proof outline

$$X_f(G) = t \Longrightarrow \quad \begin{array}{l} \exists w : V(G) \mapsto R_0^+, \text{ such that:} \\ (a) w(V) = t, \\ (b) \text{ for any independent set } I, w(I) \leq 1. \end{array}$$

Assume $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$, and let $V_k = \{v_1, \ldots, v_k\}$.
$\emptyset \neq X \subset V(G)$ is k-principal if $X \subseteq V_{k|X|}$.
X is k-sparse if X contains no k-principal subset.

Proof outline

$$X_f(G) = t \implies \begin{array}{l} \exists w : V(G) \mapsto R_0^+, \text{ such that:} \\ (a) w(V) = t, \\ (b) \text{ for any independent set } I, w(I) \leq 1. \end{array}$$

Assume $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$, and let $V_k = \{v_1, \ldots, v_k\}$.
$\emptyset \neq X \subset V(G)$ is k-principal if $X \subseteq V_{k|X|}$.
X is k-sparse if X contains no k-principal subset.

Claim

If X is k-sparse, then $w(X) \leq \frac{1}{k} w(V)$.

Lemma

There exists an orientation such that every t-principal set of G with average degree at least $8 \log t$ contains a directed cycle.

Lemma

A has weight at least $18 \log t \implies$ A contains a t-principal set with average degree at least $8 \log t$.

$H^{(m)}$, blow-up of H with *power* m: replacing each vertex by an independent set of size m.

Lemma

If $\chi(H) > x$, *and exists an orientation* D *of* $H^{(m)}$, *such that all* $K_{\lceil \frac{m}{k} \rceil, \lceil \frac{m}{k} \rceil}$ *copies in blowup of edges contains a directed cycle, then* $\chi(D) > x$.

Lemma

If $2 + \log m \leq \lceil m/x \rceil$, *and* $\chi(H) > x$, *then* $\vec{\chi}(H^{(m)}) > x$.

Theorem

$KG(nt, kt - x)$ *contains a blowup of* $KG(n, k)$ *with power* $\binom{k(t-1)}{x}$.

Theorem

If $G = KG(n, k)$, *then* $\vec{\chi}(G) \geq \left\lfloor \frac{\chi(G)}{8 \log(\chi_f(G))} \right\rfloor$.

Conjecture (Erdős-Hajnal)

For every k, l, there exists $f(k, l)$ such that any graph G with $\chi(G) \geq f(k, l)$ contains a subgraph G' of girth at least l and chromatic number at least k.

Theorem (Rödl)

For every k, there exists $f(k)$ such that any graph G with $\chi(G) \geq f(k)$ contains a triangle-free subgraph G' of chromatic number more than k.

Theorem (Mohar-W.)

Erdős-Hajnal Conjecture holds for Mycielski graphs and Kneser Graphs.

Theorem (Mohar-W.)

For every k, there exists $f(k)$ such that any graph G with $\chi_f(G) \geq f(k)$ contains a triangle-free subgraph G' of fractional chromatic number at least k.