On the Edge Spectrum of Saturation Number for Paths and Stars

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Outline

Definitions

Definition (H-saturated graphs)

A graph G is H-**saturated** if

- H is not a subgraph of G,
- for any edge $e \in E(\bar{G})$ the graph $G + e$ contains a subgraph isomorphic to H.

Definitions

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Definition (Extremal Number)

The **extremal number** $ex(n; H)$ defined as

$$ex(n; H) = \max\{|E(G)| : |V(G)| = n \text{ and } G \textit{ is } H\text{-saturated}\}$$

and the set of graphs with extremal number by $Ex(n; H)$.

Known Results

Theorem (Mantel, 1907)

$$ex(n; K_3) \leq \frac{n^2}{4}$$

and

$$Ex(n; K_3) = \{K_{\frac{n}{2},\frac{n}{2}}\}$$

The generalization of this theorem to cliques of size p is

Known Results

Theorem (Turán, 1941)

$$ex(n; K_p) \leq (1 - \frac{1}{p-1})\frac{n^2}{2}$$

and

$$Ex(n; K_p) = \{T(n, p-1)\}$$

where $T(n, p-1)$ is called the Turán Graph which is a $(p-1)$-partite graph with all parts having size as close as possible.

Definitions

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Definition (Saturation Number)

The **saturation number** $sat(n; H)$ defined as

$$sat(n; H) = \min\{|E(G)| : |V(G)| = n \text{ and } G \text{ is } H\text{-saturated}\},$$

and denote the set of graphs with saturation number by $Sat(n; H)$.

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and denote the set of graphs with saturation number by $Sat(n; H)$.

Let $SAT(n; H)$ be the set of all H-saturated graphs of order n.

Known Results

Theorem (Erdős, Hajnal, and Moon, 1964)

For $n \geq p-2$,

$$sat(n; K_p) = \binom{p-2}{2} + (n-p+2)(p-2)$$

and

$$Sat(n; K_p) = \{K_{p-2} + \bar{K}_{n-p+2}\}$$

where $+$ *denotes the join operation of graphs.*

In particular, $sat(n; K_3) = n-1$.

Some Known Results for $sat(n; H)$

- $K_{1,k}$, P_k and Matchings studied by Kászonyi and Zs. Tuza, 1986
- B_p studied by Chen, Faudree, Gould, 2008
- tK_p studied by Faudree, Ferrara, Gould, Jacobson, 2009
- Trees studied by J. Faudree, R. Faudree, Gould, Jacobson, 2009
- Cycles: $sat(n; C_n) = \frac{3n+1}{2}$, but exact value of $sat(n; C_k)$ is not known. $sat(n; C_k) = n + \frac{n}{k} + O(k^2 + \frac{n}{k^2})$, by Füredi, Kim, 2011
- A Survey of Minimum Saturated Graphs J. Faudree, R. Faudree, Schmitt, 2011

Known Results

Theorem (Kászonyi and Zs. Tuza, 1986)

Let $a_k = \begin{cases} 3.2^{p-1} - 2, & \text{if } k = 2p \\ 4.2^{p-1} - 2, & \text{if } k = 2p+1. \end{cases}$ *Then for* $n \geq a_k$ *and* $k \geq 6$, $sat(n; P_k) = n - \left\lfloor \frac{n}{a_k} \right\rfloor$ *and every graph in* $Sat(n; P_k)$ *consists of a forest with* $\left\lfloor \frac{n}{a_k} \right\rfloor$ *components.*

For $k = 2p + 2$, forest consist of binary tree T_k with root degree 3 and depth p, and for $k = 2p + 1$, it is a binary tree with double root both of which have degree 3 and depth p.

Known Results

Theorem (Kászonyi and Zs. Tuza, 1986)

Let $K_{1,k}$ denote a star on $k+1$ vertices. Then,

$$sat(n; K_{1,k}) = \begin{cases} \binom{k}{2} + \binom{n-k}{2} & \text{if } k+1 \le n \le \frac{3k}{2} \\ \left\lceil n\frac{(k-1)}{2} - \frac{k^2}{8} \right\rceil & \text{if } \frac{3k}{2} \le n \end{cases}$$

and

$$Sat(n; K_{1,k}) = \begin{cases} \{K_k \cup K_{n-k}\} & \text{if } k+1 \le n \le \frac{3k}{2} \\ \left\{G' \cup K_{\lfloor \frac{k+1}{2} \rfloor}\right\} & \text{if } \frac{3k}{2} \le n \end{cases}$$

where G' is a $(k-1)$-regular graph on $n - \left\lfloor \frac{k+1}{2} \right\rfloor$ vertices.

Known Results

Theorem

For $n \geq k+1$, $ex(n; K_{1,k}) = \left\lfloor n\frac{(k-1)}{2} \right\rfloor$, and the extremal graphs are

$$Ex(n; K_{1,k}) = \begin{cases} \{(k-1)\text{-regular graph on } n \text{ vertices}\} & \text{if } n \text{ is even or } k \text{ is odd} \\ \left\{ \begin{array}{c} \text{The graph with degree sequence} \\ k-1, k-1, ..., k-1, k-2 \end{array} \right\} & \text{otherwise.} \end{cases}$$

Known Results

In 1959, P. Erdős and T. Gallai determined the extremal number $ex(n; P_k)$ as well as the corresponding extremal graphs. We state the general version of the theorem here by Faudree and Schelp.

Theorem

For $n = l(k-1) + r$,
$ex(n; P_k) \leq l\binom{k-1}{2} + \binom{r}{2}$ *with equality if and only if* G *is either*

- *(i)* $\left(\bigcup_{i=1}^{l} K_{k-1}\right) \cup K_r$ *or*
- *(ii)* $\left(\bigcup_{i=1}^{l-t-1} K_{k-1}\right) \cup (K_{(k-2)/2} + \bar{K}_{(k+2)/2+t(k-1)+r})$ *for some* t, $0 \leq t < l$, *when* k *is even,* $l > 0$, *and* $r = k/2$ *or* $(k-2)/2$.

Known Results

In 77 Kopylov determined $ex(n; P_k)$ for connected graphs. Later on, Balister, Győri, Lehel, and Schelp obtained $ex(n; P_k)$ and also gave $Ex(n; P_k)$.

Theorem (Balister, Győri, Lehel, and Schelp, 2008)

If G is connected, then

$$ex(n; P_k) \leq \max\{\binom{k-2}{2} + (n-k+2), \binom{\lceil \frac{k}{2} \rceil}{2} + \left\lfloor \frac{k-2}{2} \right\rfloor (n - \lceil \frac{k}{2} \rceil)\}.$$

If equality occurs then G is either $G_{n,k,1}$ or $G_{n,k,\lfloor (k-2)/2 \rfloor}$.

Where $G_{n,k,s} = K_s + (K_{k-2s-1} \cup \bar{K}_{n-k+s+1})$, for $k > 2s+1$.

What is the edge spectrum for H-saturated graphs?

Definition (The Edge Spectrum for H-saturated Graphs)

The set of all values of m, where $sat(n; H) \leq m \leq ex(n; H)$, for which there exists an H-saturated graph on n vertices and m edges is called the **edge spectrum** for H-saturated graphs.

Known Results

Theorem (Barefoot, Casey, Fisher, and Fraughnaugh, 1995)

There is a K_3-saturated graph with n vertices and m edges if and only if $2n-5 \leq m \leq \frac{(n-1)^2}{4} + 1$ or $m = k(n-k)$ for some positive integer k.

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Theorem (Kinnari, Faudree, Gould and Sidorowicz, 2013)

There is a K_p-saturated graph with n vertices and m edges if and only if either

$$(p-1)\left(n-\frac{p}{2}\right)-2 \leq m \leq \frac{(p-2)n^2-2n+r(r+2)-r(p-1)}{2(p-1)}+1$$

or $m = |E(G)|$ *for some complete $(p-1)$-partite graph G on n vertices.*

Recall $sat(n;K_p) = \binom{p-2}{2}+(n-p+2)(p-2)$ by (EHM, 1964)and $ex(n;K_p) \leq (1-\frac{1}{p-1})\frac{n^2}{2}$ by (Turan, 1941).

Known Results

Theorem (Gould, Tang, Wei, Zhang, 2012)

There are P_5-saturated graphs with n vertices and m edges provided $sat(n; P_5) \leq m \leq ex(n; P_5)$, except in the cases

$$m \in \begin{cases} \left\{\frac{3n-5}{2}\right\} & \text{if } n \equiv 3 \pmod 4, \\ \left\{\frac{3n}{2}-3, \frac{3n}{2}-2, \frac{3n}{2}-1\right\} & \text{if } n \equiv 0 \pmod 4. \end{cases}$$

Note that $sat(n; P_5) = \frac{5n-4}{6}$) by (KT, 1986) and $ex(n; P_5) = \frac{3n}{2}$ by (EG, 1959).

Known Results

Theorem (Gould, Tang, Wei, Zhang, 2012)

For $n \geq 10$ and $(n, m) \neq (11, 14)$, there are P_6-saturated graphs with n vertices and m edges provided $sat(n; P_6) \leq m \leq ex(n; P_6)$, except in the cases

$$m \in \begin{cases} \{2n-4, 2n-2, 2n-1\} & \textit{if } \; n \equiv 0 \pmod 5, \\ \{2n-4\} & \textit{if } \; n \equiv 2, 4 \pmod 5. \end{cases}$$

Note that $sat(n; P_6) = \frac{9n}{10}$) by (KT, 1986) and $ex(n; P_6) = 2n$ by (EG, 1959).

Our Main Results

Theorem

Let $\epsilon > 0$, and let k and n be integers with $k \geq k_0(\epsilon)$ and $n \geq a_k$, where a_k is defined previously. Then for any integer m such that $sat(n; P_k) \leq m \leq ex(n; P_k) - (\sqrt{2} + \epsilon)k^{3/2}$ there exists a P_k-saturated graph on n vertices with m edges.

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We also show that $ex(n; P_k) - (\sqrt{2} + o(1))k^{3/2}$ is the best possible upper bound, up to the constant $\sqrt{2}$. More precisely, we show that for each sufficiently large k there exists an infinite sequence of n and m with $sat(n; P_k) \leq m \leq ex(n; P_k) - \epsilon k^{3/2}$, and no P_k-saturated graph exists with n vertices and m edges.

Our Main Results...

Theorem

Let n and k be two integers such that $n \geq k \geq 1$. Then for any integer m such that $sat(n; K_{1,k}) \leq m \leq ex(n; K_{1,k})$ there is a $K_{1,k}$-saturated graph on n vertices with m edges.

Edge Spectrum of Paths

We use the following Lemma to get the top part of the edge spectrum for paths...

Lemma

Let $f(n)$ be the largest integer such that every integer between 0 and $f(n)$ can be represented as $\sum_{i\geq 1}\binom{r_i}{2}$ with $\sum_{i\geq 1} r_i = n$ and integers $r_i \geq 0$. Then $f(n) \geq \frac{1}{2}(n-2\sqrt{n})^2$ for $n \geq 2$.

Sketch of the Proof

Part 1. Let $n = l(k-1) + r$, $0 \le r < k-1$. We will deal with the cases when r is large and r is small separately.

Case 1.1: $\binom{r}{2} \ge k-2$.

Let

$$G_0 = \left(\bigcup_{i=1}^{l-s} K_{k-1}\right) \cup \left(\bigcup_{i=1}^{s-1} K_{k-2}\right) \cup H \cup K_r,$$

where $H = K_1 + \left(K_{k-3} \cup \bar{K}_s\right)$.

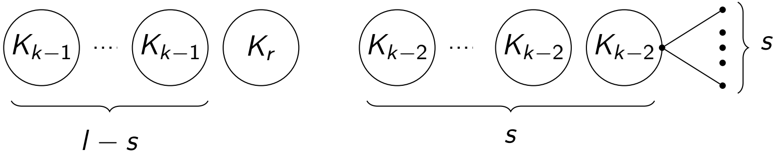

Figure: G_0

Then $|E(G_0)| = e - s(k-3)$, where $e = ex(n; P_k)$.

Sketch of the Proof

Case 1.2: $\binom{r}{2} < k - 2$. In this case we let

$$G_1 = \left(\bigcup_{i=1}^{l-s} K_{k-1}\right) \cup \left(\bigcup_{i=1}^{s-1} K_{k-2}\right) \cup H_{r+s},$$

where $H_{r+s} = K_1 + \left(K_{k-3} \cup \bar{K}_{s+r}\right)$. Note that this graph is saturated provided $r + s \geq 2$, and $|E(G_1)| = e - s(k-3) - a$, where $e = ex(n; P_k)$ and $a = \binom{r}{2} - r$.

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The following claim tells us that by moving vertices from some cliques to other cliques in G_0 (or in G_1) we gain some edges.

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The following claim tells us that by moving vertices from some cliques to other cliques in G_0 (or in G_1) we gain some edges.

Claim: Replacing the cliques K_{k-2} by K_{k-1-r_i} in graph G_0 so that their total order remains constant always gains $\sum \binom{r_i}{2}$ edges, where $r_i \geq 0$ are such that $\sum_{i \in I} r_i = |I|$ is the original number of K_{k-2} cliques.

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Proof of Claim: Replacing K_{k-2} by K_{k-1-r_i} changes the number of edges by

$$\begin{aligned}\binom{k-1-r_i}{2} - \binom{k-2}{2} &= \frac{1}{2}\big((k-1-r_i)(k-2-r_i) - (k-2)(k-3)\big) \\ &= \frac{1}{2}\big(r_i^2 - (2k-3)r_i + 2(k-2)\big) = (k-2)(1-r_i) + \binom{r_i}{2}.\end{aligned}$$

Summing over i and noting that $\sum(1 - r_i) = 0$ gives the result.

Part2: Define *Forming a Pendant Triangle* at a vertex x as follow: remove two vertices from the s pendant vertices and form a triangle whose vertices are the vertex x and two removed vertices. By Forming a Pendant Triangle at a vertex x we gain exactly 1 edge, and the resulting graph is still P_k-saturated

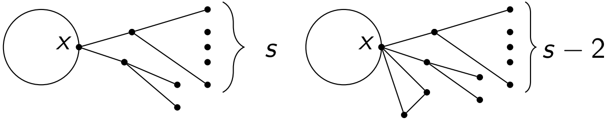

Figure: Forming a Pendant Triangle at x

Part 3: We start with a smallest saturated graph G, which is a forest consisting of almost binary trees T_k, with $|E(G)| = sat(n; P_k) = n - \left\lfloor \frac{n}{a_k} \right\rfloor$. By forming pendant triangles in a tree component many times, we cover the bottom part of the spectrum.

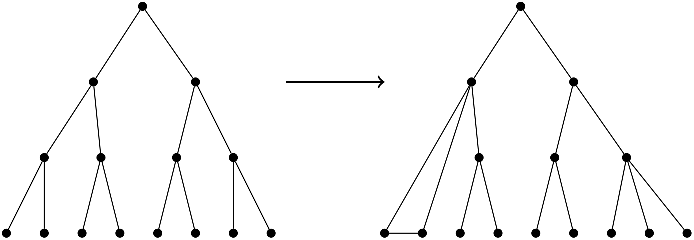

Figure: Forming a Pendant Triangle on the bottom level

There is a gap near $ex(n : P_k)$

Corollary

Let k be sufficiently large, and let $n = (k-1)l$. Then there is an integer $\beta_0 \sim k^{3/2}/\sqrt{24}$ such that there is no P_k-saturated graph of size $ex(n; P_k) - \beta_0$.

The End

Thank You!