Homology of Filters in the Partition Lattice

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Example

$123 - 47 - 6 - 5 \prec 123 - 467 - 5$

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- Wachs then showed $\Pi_{n,d}$ is EL shellable, gave explicit basis for top homology of $\triangle(\Pi_{n,d})$

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- We would like these general ideas to simplify to cases already studied, like the d-divisible partition lattice $\Pi_{n,d}$.
- We will do this by working with *ordered set partitions*, which have nicer structure, then translate information back to ordered set partitions with the *forgetful* map.

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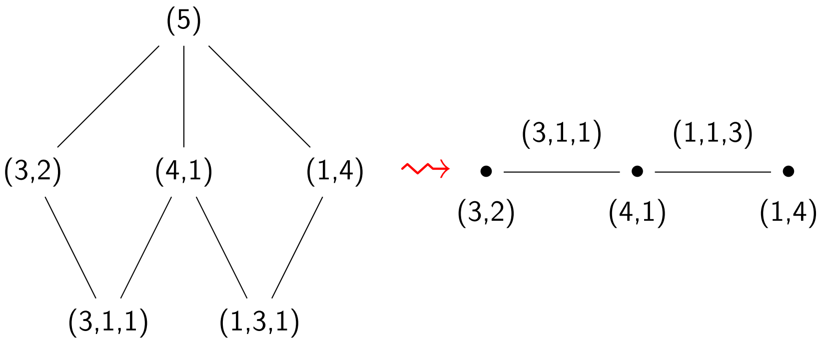
$1-234-5$ and $5-1-234$ are distinct ordered set partitions in Q_5.

a filter $F \subseteq \mathrm{Comp}(n)$ as a simplicial complex

- Let F be a filter in $\mathrm{Comp}(n)$, so if $\vec{d} \in F$, then if $\vec{f} > \vec{d}$, then $\vec{f} \in F$.
- In the dual poset $\mathrm{Comp}(n)^*$ we view F as a simplicial complex, which we call Δ, since upper order ideals in $\mathrm{Comp}(n)$ are boolean.

$\langle(3,1,1),(1,3,1)\rangle \subseteq \mathrm{Comp}(5)$ as a simplicial complex



Construction of Q_Δ^*

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Let $\text{type} : Q_n \longrightarrow \text{Comp}(n)$ be defined as
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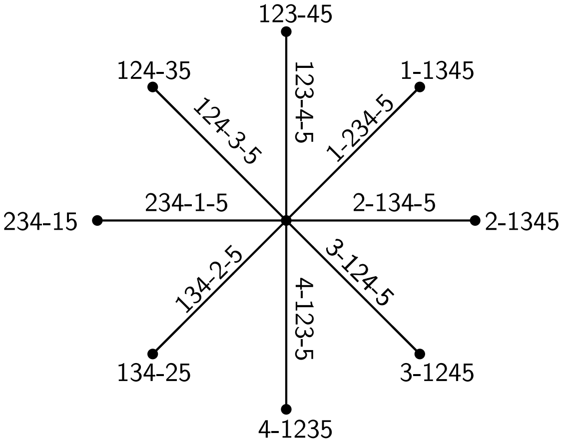
Definition

Let $Q^*_\Delta \subseteq Q_n$ be the collection of all ordered set partitions whose type is in $\Delta \subseteq \text{Comp}(n)$, and with n in the last block.

Topology of Q_Δ^*

- We view Q_Δ^* as a simplicial complex under the dual order–go down by merge, up by split.
- The *link* of a face σ in Q_Δ^* is defined to be the collection of faces in Q_Δ^* containing σ.

Q^*_Δ for $\Delta = \langle(3,1,1),(1,3,1)\rangle \subseteq \mathrm{Comp}(5)$



Note: The center vertex is $1234 - 5$

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- Let Π^*_Δ in Π_n be the image of Q^*_Δ under f
- f induces a homotopy equivalence between $\triangle(\Pi_\Delta - \{\hat{1}\})$ and Q^*_Δ using the Quillen Fiber Lemma.

Topology of Q^*_Δ continued

Theorem (Ehrenborg, Hedmark, 2015)

The ith reduced homology group of Q^*_Δ is given by

$$\widetilde{H}_i(Q^*_\Delta) \cong_{\mathfrak{S}_{n-1}} \bigoplus_{\vec{c} \in \Delta} \widetilde{H}_{i-|\vec{c}|+1}(\mathrm{lk}_{\vec{c}}(\Delta)) \otimes S^{B^*(\vec{c})}$$

Where $S^{B^*(\vec{d})}$ is the border strip Specht module on the shape $(d_1, \ldots, d_k - 1)$

- Main Idea: We can translate topological data from Δ to Q^*_Δ, and Δ is a smaller complex.

$\Delta = \langle \vec{d} \rangle$, an example

Example

- Consider $\langle \vec{d} \rangle \subseteq \text{Comp}(n)$, with $|\vec{d}| = k$.
- $\Delta_{\vec{d}}$ is then a simplex with facet $\vec{d}$.
- $\Delta_{\vec{d}}$ is contractible, so $\text{lk}_{\vec{c}}(\Delta_{\vec{d}})$ is contractible for all $\vec{c} \neq \vec{d}$ in the simplex.
- $\text{lk}_{\vec{d}}(\Delta_{\vec{d}}) = \emptyset$, with $\widetilde{H}_{-1}(\emptyset) = \mathbb{C}$.
- Only term that survives is when $\vec{c} = \vec{d}$ and $i - k + 1 = -1$, so $i = k - 2$.
- $\widetilde{H}_{k-2}(Q^*_{\Delta_{\vec{d}}}) \cong S^{B^*(\vec{d})}$

$\Delta = \langle (d, d, \ldots, d) \rangle$, The d-divisible partition lattice

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- $\Pi_\Delta = f(Q_\Delta^*)$ is the collection of all partitions with all block sizes divisible by d, hence $\Pi_\Delta = \Pi_{n,d}$

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- $\Pi_\Delta = f(Q^*_\Delta)$ is the collection of all partitions with all block sizes divisible by d, hence $\Pi_\Delta = \Pi_{n,d}$
- $\triangle(\Pi_{n,d} - \{\hat{1}\}) \cong_{\mathfrak{S}_{n-1}} S^{B^*((d,d,\ldots,d))}$, which recovers Calderbank, Hanlon, Robinson result

Further Directions

- The last example shows our construction Π_Δ simplifies to previously known cases, such as the d-divisible partition lattice and the sub-lattice where all block sizes must be at least k.
- Is there a nice way to compute the $\mathfrak{S}_n$ action on $\widetilde{H}_i(\Pi_\Delta - \{\hat{1}\})$?
- We can apply the Frobenius characteristic ch to convert representations into symmetric functions. Can we see any of our result using plethysm?

The End

Thank you for your time