Total domination polynomials of graphs

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History

The domination problem was studied from the 1950s onwards. But a few research papers on domination were published between 1958 and 1975. A survey paper by Cockayne and Hedetniemi (1975) served to focus attention on the subject. After 1975, the rate of research on domination significantly increased. And the concept of connected dominating set, independent dominating set and total dominating set were introduced afterwards. In the last decade, the field of domination polynomials is a considerable research topic. One can obtain many interesting combinatorial and graphic information of a graph from its domination polynomial. In 2012, A. Vijayan and S. S. Kumar introduced total domination polynomial of a graph.

Definitions

Let $G = (V, E)$ be a simple graph, $S \subseteq V(G)$ and $u, v \in V(G)$.

- $N_G(v) = \{w \in V(G) : vw \in E(G)\}$, $N_G[v] = N_G(v) \cup \{v\}$
- $G - S$ is the subgraph induced by $V(G) - S$
- G/v is the contracted graph by the removal of v and the addition of edges between any pair of non-adjacent neighbors of v
- $G \ominus u$ or $G \ominus u \ominus v$ represents a subgraph induced by $V(G) - N_G[u]$ or $V(G) - N_G[u] - N_G[v]$ respectively.

Definitions

A vertex set D of a graph G is a total dominating set if every vertex of $V(G)$ is adjacent to some vertices of D. Let $\mathcal{D}_t(G, i)$ be the set of all total dominating sets of size i and set $d_t(G, i) = |\mathcal{D}_t(G, i)|$. The total domination polynomial of G is defined as

$$D_t(G, x) = \sum_{i=1}^{|V(G)|} d_t(G, i)x^i$$

Observation

Let G, P_n and C_n be a graph, a path and a cycle with n vertices. Then
(i) $D_t(G,x) \neq 0$ if and only if G has no isolated vertices.
(ii) The number of supporting vertices of G is $n - d_t(G, n-1)$.
(iii) [Chaluvaraju, 2014] $D_t(G,x) = D_t(G_1,x)D_t(G_2,x)$, *where* $G = G_1 \cup G_2$.
(iv) $D_t(P_1,x) = 0, D_t(P_2,x) = x^2, D_t(P_3,x) = x^3 + 2x^2, D_t(P_4,x) = x^4 + 2x^3 + x^2$.
(v) [Vijayan, 2012] $D_t(C_3,x) = x^3 + 3x^2, D_t(C_4,x) = x^4 + 4x^3 + 4x^2, D_t(C_5,x) = x^5 + 5x^4 + 5x^3, D_t(C_6,x) = x^6 + 6x^5 + 9x^4$.

For general graph, the problem of finding total domination number is NP-completeness. It will be not be easy to find the total domination polynomial for general graph G.

For any graph G, let $W \subseteq V(G)$ and C_W be a statement on W. Denote $D_t(G,x)\{C_W\}$ to be the generating function for the number of total dominating sets of G under the condition C_W:

$$D_t(G,x)\{C_W\} = \sum_{N_G[W]=V(G)} \varphi(W)x^{|W|},$$

$$\text{where } \varphi(W) = \begin{cases} 1, & \text{if } C_W \text{ holds for } W, \\ 0, & \text{otherwise.} \end{cases}$$

For instance, if $w \in V(G)$, then $D_t(G,x)\{w \notin W\}$ represents the total domination polynomial generated by all total domination sets W of G with $w \notin W$.

Reduction-formulas of total domination polynomials

Theorem

For any connected graph G and $u \in V(G)$,

$$\begin{aligned} D_t(G,x) &= D_t(G-u,x) + xD_t(G/u,x) \\ &\quad -(1+x)D_t(G/u,x)\{N_G(u) \cap W = \emptyset\} \\ &\quad + \sum_{v \in N_G(u)} x^2 1_{G \ominus u \ominus v}, \end{aligned}$$

where,

$$1_{G \ominus u \ominus v} = \begin{cases} 1, & \text{if } G \ominus u \ominus v = \phi, \\ D_t(G \ominus u \ominus v, x), & \text{if } G \ominus u \ominus v \neq \phi. \end{cases}$$

Corollary

For any connected graph G, $u \in V(G)$, if either (i) there exist $v \in V(G)$ such that $N_G[v] \subseteq N_G[u]$ or (ii) there exists a supporting vertex $v \in N_G(u)$, then

$$D_t(G,x) = D_t(G-u,x) + xD_t(G/u,x) + \sum_{v \in N_G(u)} x^2 1_{G \ominus u \ominus v}.$$

Reduction-formulas of total domination polynomials

Theorem

Let P_n be a path of order $n \geq 5$,

$$D_t(P_n, x) = xD_t(P_{n-1}, x) + x^2 D_t(P_{n-3}, x) + x^2 D_t(P_{n-4}, x).$$

Reduction-formulas of total domination polynomials

Corollary

Let P_n be a path of order $n \geq 1$,

$$D_t(P_n, x) = \begin{cases} \frac{2x^{\frac{n}{2}}}{x+4} + p(n), & \text{if} \quad n = 4m, \\ -\frac{(x^2+3x)x^{\frac{n-1}{2}}}{x+4} + p(n), & \text{if} \quad n = 4m+1, \\ -\frac{2x^{\frac{n}{2}}}{x+4} + p(n), & \text{if} \quad n = 4m+2, \\ \frac{(x^2+3x)x^{\frac{n-1}{2}}}{x+4} + p(n), & \text{if} \quad n = 4m+3, \end{cases}$$

where $p(n) = \frac{(x+2-\sqrt{x^2+4x})(x-\sqrt{x^2+4x})^n+(x+2+\sqrt{x^2+4x})(x+\sqrt{x^2+4x})^n}{2^{n+1}(x+4)}$.

Reduction-formulas of total domination polynomials

Theorem

For any connected graph G and $e = uv \in E(G)$,

$$\begin{aligned} D_t(G,x) &= D_t(G-e,x) + x^2 1_{G \ominus u \ominus v} \\ &+ (1+x)[D_t(G-e \ominus u, x)\{v \in W\} \\ &+ D_t(G-e \ominus v, x)\{u \in W\}]. \end{aligned}$$

Theorem

For any cycle C_n with $n \geq 7$,

$$D_t(C_n, x) = xD_t(C_{n-1}, x) + x^2 D_t(C_{n-3}, x) + x^2 D_t(C_{n-4}, x).$$

Reduction-formulas of total domination polynomials

Corollary

For any cycle C_n with $n \geq 3$ vertices,

$$D_t(C_n, x) = \begin{cases} 2(-x)^{\frac{n}{2}} + q(n), & \text{if} \quad n = 2m, \\ q(n), & \text{if} \quad n = 2m+1, \end{cases}$$

where $q(n) = \frac{(x-\sqrt{x^2+4x})^n + (x+\sqrt{x^2+4x})^n}{2^n}$.

Theorem

Let T_n be a tree in $\mathcal{T}_n$ that is a class of trees with n vertices.
(i) $d_t(T_n, i) \leq \binom{n-1}{i-1}$ and the equality holds if and only if $T_n = S_n$.
(ii) There is no tree T' in $\mathcal{T}_n$ such that $d_t(T_n, i) \geq d_t(T', i)$ for each $i \geq 2$.

In $D_t(G, x)$, let's set $x = -1$.

$$D_t(G, -1) = \sum_{i=1}^{|V(G)|} d_t(G, i)(-1)^i.$$

This tells us the difference between the number of total dominating sets with odd and even size.

Theorem

Let S_n, P_n be a star and a path with n vertices. Then
(i) $D_t(S_n, -1) = 1$;
(ii) $D_t(P_{n_0}, -1) = D_t(P_{n_2}, -1) = D_t(P_{n_3}, -1) = D_t(P_{n_5}, -1) = 1$,
$D_t(P_{n_1}, -1) = D_t(P_{n_4}, -1) = 0$, *where* $n_i = i (mod\, 6)$ *and* $i = 0, 1, ..., 5$.

Theorem

Let F_n be any forest with n vertices, then $D_t(F_n, -1) \in \{0, 1\}$

Theorem

If G is a connected graph of order n, then the number of vertices of degree 2 is at least $\binom{n}{2} - \binom{r}{2} - r(n-r) - d_t(G, n-2)$, where r is the number of supporting vertices in G.


thank you