# On Sectional Complements in Lifted Groups

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Joint work with Rok Požar

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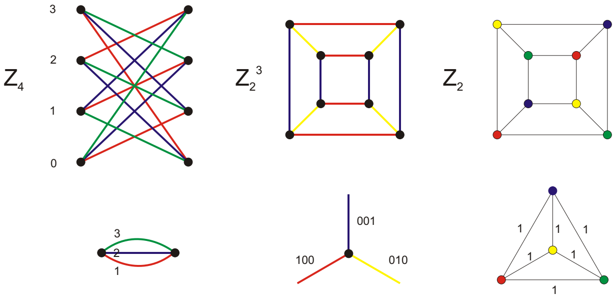

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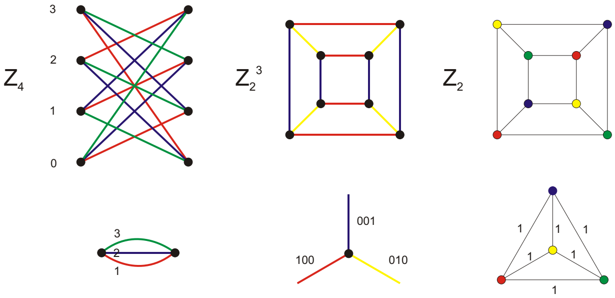


**Construction/reconstruction**
by a **regular voltage function** $\zeta\colon A(X) \to \Gamma \cong \mathrm{CT}$

# Lifting automorphisms along regular covering projections

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**Djoković, 1974:** $G$ $s$-**arc transitive** $\Rightarrow$ $\tilde{G}$ $s$-**arc transitive**

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Rok Požar, 2013: Magma packge, available on the Web

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Feng, Kutnar, M., Marušič, On 2-fold covers of graphs, JCTB (2008)

M., Požar, On split lifts with sectional complements, submitted.

# Sectional split lifts: characterization via stabilizers

**Thm.** A complement $\bar{\mathrm{G}}$ is sectional over $\Omega = \mathrm{G}(\mathrm{b})$ if and only if the intersection $\bar{\mathrm{G}} \cap \widetilde{\mathrm{G}_{\mathrm{b}}}$ is the stabilizer of some $\tilde{\mathrm{u}} \in \mathrm{fib}_{\mathrm{b}}$.

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**Derivations**

$$\mathrm{Der}(\mathrm{G}, \mathrm{CT}) = \{\delta \colon \mathrm{G} \to \mathrm{CT} \mid \delta_{\mathrm{gh}} = \delta_\mathrm{g} \theta_\mathrm{g}(\delta_\mathrm{h})\}$$

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**Thm**. Suppose that $G$ lifts as a sectional split extension over $\Omega = G(b)$. Then

Sectional complements $\leftrightarrow \{\delta \in \mathrm{Der}(G, CT) \mid \delta|_{G_b} \in \mathrm{Inn}(G_b, CT)\}$.

$CT$ **abelian** – sectional complements/conj $\leftrightarrow \leq H^1(G, CT)$

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**Thm**. Let $\mathcal{C}$ be a conjugacy class of a sectional complements over $\mathrm{G(u)}$. Then the number of invariant sections through $\tilde{\mathrm{u}} \in \mathrm{fib_b}$ that belong to some sectional complement from $\mathcal{C}$ does not depend on $\tilde{\mathrm{u}} \in \mathrm{fib_b}$, and is equal to

$$|\mathrm{Fix}(\tilde{\mathrm{G}}_{\tilde{\mathrm{u}}})| \,/\, |\mathrm{C}_{\mathrm{CT_p}}(\bar{\mathrm{G}})| = |\mathrm{C}_{\mathrm{CT_p}}(\tilde{\mathrm{G}}_{\tilde{\mathrm{u}}})| \,/\, |\mathrm{C}_{\mathrm{CT_p}}(\bar{\mathrm{G}})|.$$

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**Note**.
We can explicitly find all $\mathbb{Z}_p$-elementary abelian regular coverings along which $G$ lifts in this manner. The problem is reduced to finding invariant subspaces of matrix group linearly representing the action of $G$ on the first homology group $H_1(X, \mathbb{Z}_p)$. (M, Marušič, Potočnik, JACO 2004).

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$G$ lifts along a regular covering projection $p \colon \tilde{X} \to X$ as a sectional split extension over $\Omega$ if and only if there exists an automorphism $g^{\sharp_\Omega} : \Gamma \to \Gamma$

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**Note.** For $\Omega = V(X)$ we get Biggs' compatibility condition, AGT 1974.

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**Note**.
Finding the right voltage assignment is exponentially difficult ! However, for **abelian** (**solvable**) covers **there is an efficient algorithm**.

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**Note.**

- Symbolic computation can be avoided, and can be carried out over $\mathbb{Z}$.
- All solutions/def. rel. for $\Gamma \leftrightarrow$ All sectional complements.
- If $\mathrm{CT} = \mathbb{Z}_p^r$, then $\mathcal{O}(r^3 V(X)^4)$.
- $\mathrm{CT}$ is solvable – A series of elementary abelian covers.

**Thank you!**