The Euler-Mahonian Identity

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November 14, 2015

What is the Euler-Mahonian Identity

Theorem (L. Carlitz, 1975)

$$\sum_{k\geq 0}[k+1]_q^n t^k = \frac{\sum_{\pi\in S_n} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}}{\prod_{j=0}^{n}(1-tq^j)}$$

where $\mathrm{des}(\pi) = \#\mathrm{Des}(\pi) = \#\{i|\pi_i > \pi_{i+1}\}$, $\mathrm{maj}(\pi) = \sum_{j\in\mathrm{Des}(\pi)} j$, *and* $[k+1]_q = 1+q+q^2+\cdots+q^k$.

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5 "newer" proofs:

- R. Adin, F. Brenti, and Y. Roichman (2004): Descent basis representatives over the coinvariant algebra for S_n and a Hilbert series calculation
- M. Beck and B. Braun (2013): Polyhedral geometry of $\mathrm{cone}([0,1]^n)$
- T. Pensyl and C. Savage (2012): Generating functions for lecture hall partitions
- J. Shareshian and M. Wachs (2010): Eulerian quasisymmetric function identities
- J. Huang (2015): Stanley-Reisner rings and 0-Hecke algebra representations.

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Definition

The *coinvariant algebra* of S_n is P_n/I_n.

For any $\pi \in S_n$,

$$a_\pi = \prod_{j \in \mathrm{Des}(\pi)} x_{\pi(1)} \cdots x_{\pi(j)}$$

The *Garsia-Stanton descent basis* for P_n/I_n is

$$\{a_\pi + I_n : \pi \in S_n\}$$

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Short Term Goal # 1: What is the representative of a given monomial in terms of these cosets?

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- The *complementary partition* of m, $\mu = \mu(m)$, is the partition $\mu = (p_{\pi(1)} - d_1(\pi), \cdots, p_{\pi(n)} - d_n(\pi))'$.

$$\mu_j(m) = \#\{i : p_{\pi(i)} - d_i(\pi) \geq j\}$$

where $d_i(\pi) = \#\{j \in \mathrm{Des}(\pi) : i \leq j\}$

If $m_1, m_2 \in P_n$ have the same total degree, then we say $m_1 \prec m_2$ if

- $\lambda(m_1) \lhd \lambda(m_2)$ (Strict dominance order)
- $\lambda(m_1) = \lambda(m_2)$ and $\mathrm{inv}(\pi(m_1)) > \mathrm{inv}(\pi(m_2))$

Theorem (Adin-Brenti-Roichman, 2004)

Let $m \in P_n$ be a monomial and let $\pi = \pi(m)$ and $\mu = \mu(m)$. Let S be the set of monomials which appear in $a_\pi e_\mu$. Then

1. $m \in S$
2. *if $m' \in S$ and $m' \neq m$, then $m' \prec m$*

Lemma (Adin-Brenti-Roichman, 2004)

(Straightening Lemma)
Each monomial $m \in P_n$ can be written as

$$m = e_{\mu(m)} a_{\pi(m)} + \sum_{m' \prec m} n_{m,m'} e_{\mu(m')} a_{\pi(m')}$$

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Short Term Goal # 2: We want a Hilbert series for P_n.

Let $P_\lambda^{\trianglelefteq} = \mathrm{span}_{\mathbb{C}}\{m : \ \lambda(m) \trianglelefteq \lambda\}$.

A filtration of P_n

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Claim: This forms a filtration of P_n, as

$$P_n = \bigcup_\lambda P_\lambda^{\trianglelefteq}$$

and

$$P_{\lambda_1}^{\trianglelefteq} \cdot P_{\lambda_2}^{\trianglelefteq} \subseteq P_{\lambda_1+\lambda_2}^{\trianglelefteq}$$

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Claim: $\lambda(a_\pi) = \lambda_{\mathrm{Des}(\pi)} = (d_1(\pi), \cdots, d_n(\pi))$

Theorem (Adin-Brenti-Roichman, 2004)

Let $n \in \mathbb{P}$

$$\sum_{\ell(\lambda)\leq n} \binom{n}{\bar{m}(\lambda)} \prod_{i=1}^{n} q_i^{\lambda_i} = \frac{\sum_{\pi\in S_n} \prod_{i=1}^{n} q_i^{d_i(\pi)}}{\prod_{i=1}^{n}(1 - q_1 \cdots q_i)}$$

where $\binom{n}{\bar{m}(\lambda)} = \binom{n}{m_0(\lambda), m_1(\lambda), m_2(\lambda), \ldots}$ *and* $m_j(\lambda) = \#\{i : \lambda_i = j\}$

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Lemma (Adin-Brenti-Roichman, 2004)

The mapping $m \mapsto (\pi(m), \mu(m)')$ is a bijection between monomials in P_n, and pairs $(\pi, \tilde{\mu})$, where $\pi \in S_n$ and $\tilde{\mu}$ is a partition with at most n parts.

Sketch of Proof

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- RHS: Since $\lambda(m) = \lambda(a_{\pi(m)}) + \mu(m)'$ by the lemma we have

$$\sum_{m \in P_n} \bar{q}^{\lambda(m)} = \sum_{m \in P_n} \bar{q}^{\lambda(a_{\pi(m)}) + \mu(m)'} = \sum_{\pi \in S_n} \bar{q}^{\lambda(a_{\pi(m)})} \cdot \sum_{\mu} \bar{q}^{\mu}$$

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This gives

$$\frac{\sum_{\pi \in S_n} \bar{q}^{\lambda_{\mathrm{Des}(\pi)}}}{\prod_{i=1}^{n} (1 - q_1 \cdots q_i)}$$

Proof of Euler-Mahonian Identity

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The LHS gives

$$\sum_{\ell(\lambda) \leq n} \binom{n}{\bar{m}(\lambda)} q^{\sum_i \lambda i} \cdot \frac{t^{\lambda_1}}{1 - t} = \sum_{k=0}^{\infty} \sum_{\substack{\ell(\lambda) \leq n \\ \lambda_1 \leq k}} \binom{n}{\bar{m}(\lambda)} q^{\sum_i \lambda_i} t^k = \sum_{k=0}^{\infty} [k+1]_q^n t^k$$

Research Plan

Stage 1: Applying these methods to the cone over the unit cube.

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Additionally, we have proposed candidates for analogues of

- elementary symmetric functions
- "descent" basis elements

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$$L_n := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \frac{\lambda_3}{3} \leq \cdots \leq \frac{\lambda_n}{n} \right\}$$

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This cone has semigroup algebra

$$\mathcal{L}_n := \mathbb{C}\left[x^{v^A} : A \subseteq [n-1] \right]$$

where $v^A = (0, 0, \cdots, 0, i_1, i_2, \cdots, i_k, i_k + 1)$ with $A = \{i_1 < i_2 < \cdots < i_k\}$

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$A = \{i_1 < i_2 < \cdots < i_k\}$
We have a proposed candidate for an analogue to elementary symmetric functions.

Thank You for Listening!