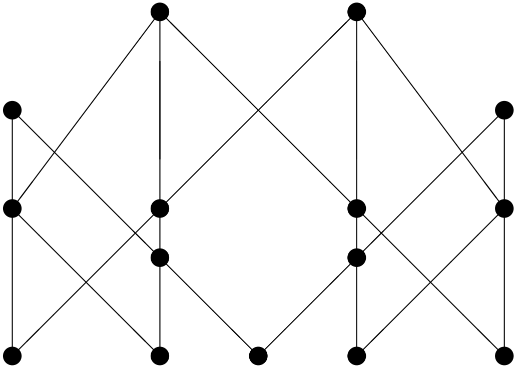
A polynomial algorithm to check the fixed point property for ordered sets of dimension 2

Bernd Schröder

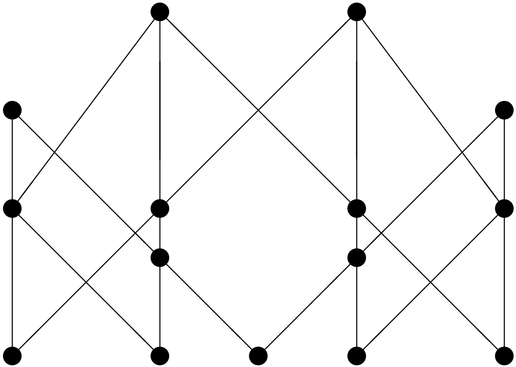
Definition.

Definition. *An* **ordered set** *is a pair* $(P, \leq)$ *of a set* P *and a reflexive, antisymmetric and transitive relation* $\leq$*, the order relation.*

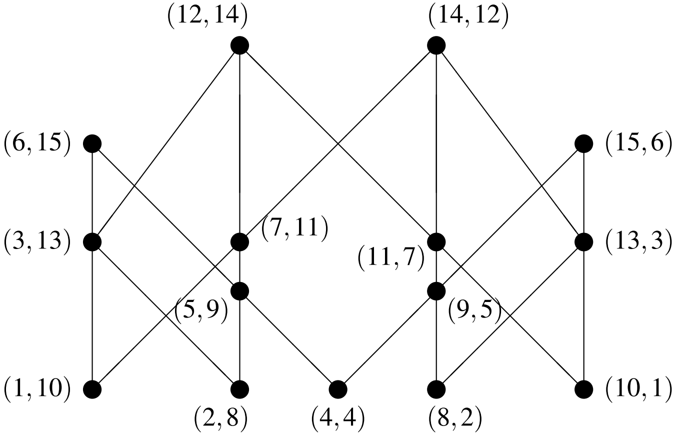
Definition. *An* **ordered set** *is a pair* $(P, \leq)$ *of a set* P *and a reflexive, antisymmetric and transitive relation* $\leq$*, the order relation.*



Definition. *A finite ordered set P is* **two-dimensional** *iff its order is the intersection of two linear orders.*



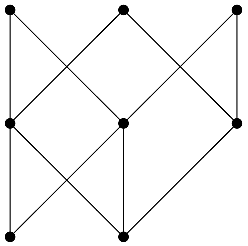
Definition. *A finite ordered set P is* **two-dimensional** *iff its order is the intersection of two linear orders.*



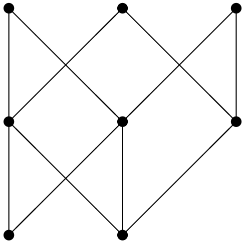
Definition.

Definition. *A function from a finite ordered set* $(P,\leq)$ *to another ordered set* $(Q,\leq)$ *is called* **order-preserving** *iff, for all* $x,y\in P$ *we have that* $x\leq y$ *implies* $f(x)\leq f(y)$.

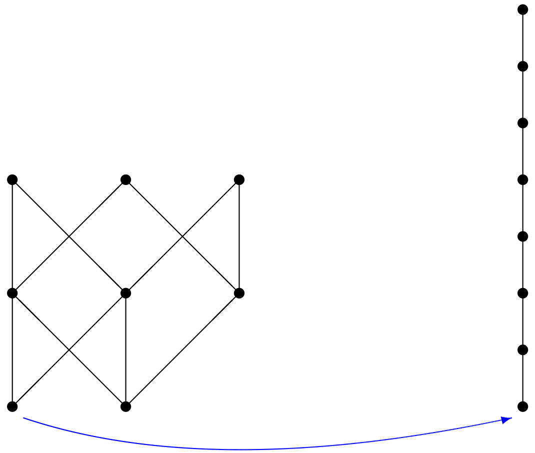
Definition. *A function from a finite ordered set* $(P, \leq)$ *to another ordered set* $(Q, \leq)$ *is called* **order-preserving** *iff, for all* $x, y \in P$ *we have that* $x \leq y$ *implies* $f(x) \leq f(y)$.



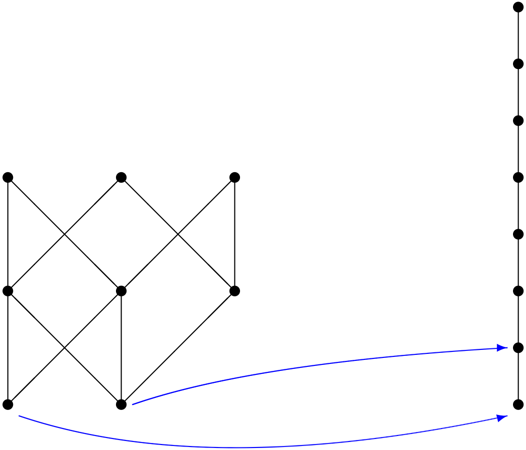
Definition. *A function from a finite ordered set $(P,\leq)$ to another ordered set $(Q,\leq)$ is called* **order-preserving** *iff, for all $x,y \in P$ we have that $x \leq y$ implies $f(x) \leq f(y)$.*



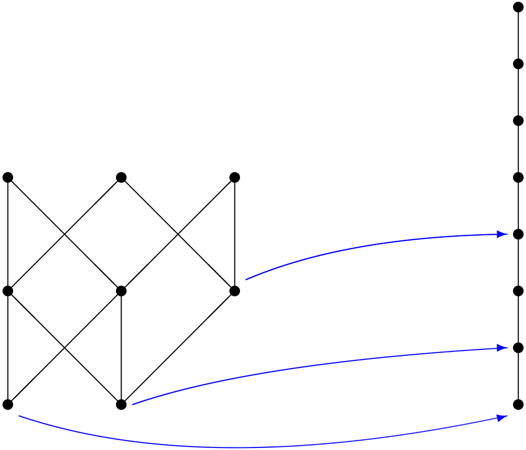
Definition. *A function from a finite ordered set* $(P,\leq)$ *to another ordered set* $(Q,\leq)$ *is called* **order-preserving** *iff, for all* $x,y\in P$ *we have that* $x\leq y$ *implies* $f(x)\leq f(y)$.



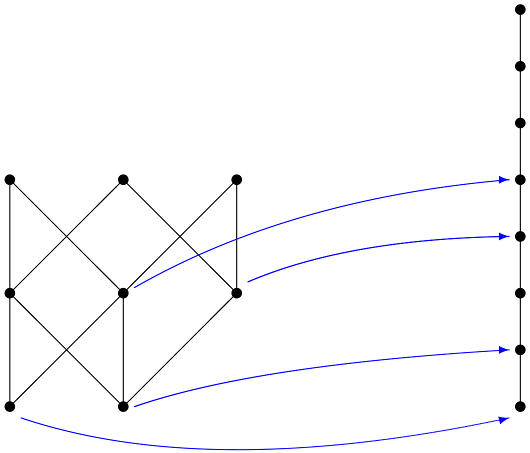
Definition. *A function from a finite ordered set* $(P,\leq)$ *to another ordered set* $(Q,\leq)$ *is called* **order-preserving** *iff, for all* $x,y \in P$ *we have that* $x \leq y$ *implies* $f(x) \leq f(y)$.



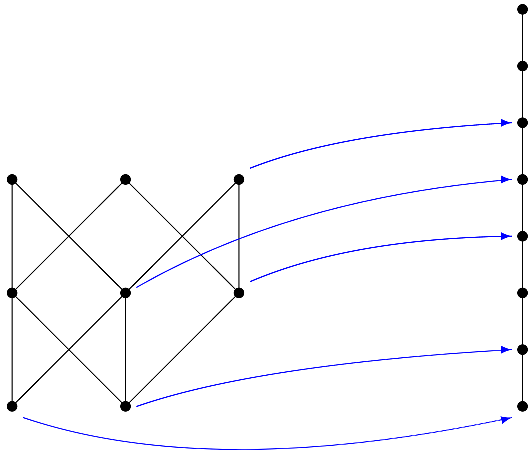
Definition. *A function from a finite ordered set* $(P, \leq)$ *to another ordered set* $(Q, \leq)$ *is called* **order-preserving** *iff, for all* $x, y \in P$ *we have that* $x \leq y$ *implies* $f(x) \leq f(y)$.



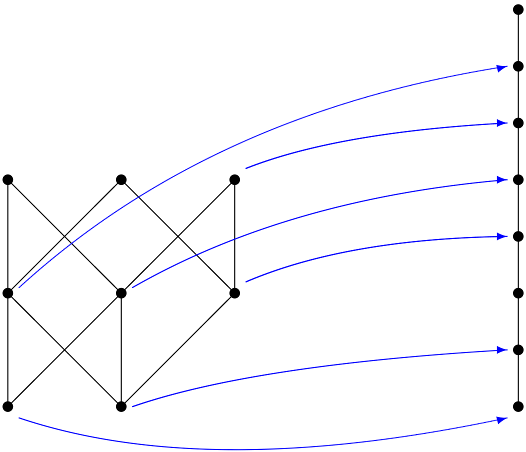
Definition. *A function from a finite ordered set $(P,\leq)$ to another ordered set $(Q,\leq)$ is called* **order-preserving** *iff, for all $x,y\in P$ we have that $x\leq y$ implies $f(x)\leq f(y)$.*



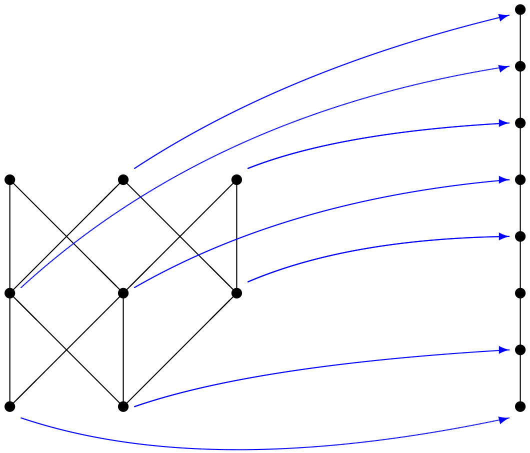
Definition. *A function from a finite ordered set* $(P, \leq)$ *to another ordered set* $(Q, \leq)$ *is called* **order-preserving** *iff, for all* $x, y \in P$ *we have that* $x \leq y$ *implies* $f(x) \leq f(y)$.



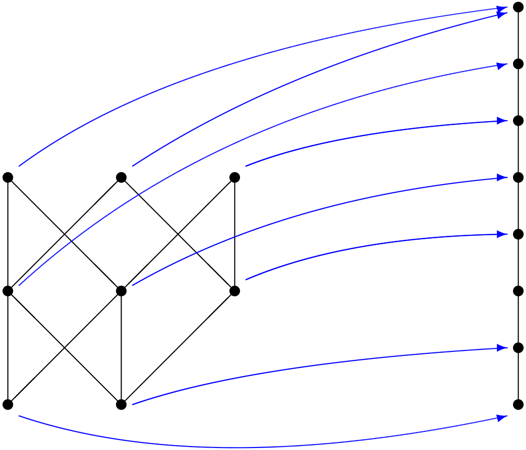
Definition. *A function from a finite ordered set* $(P,\leq)$ *to another ordered set* $(Q,\leq)$ *is called* **order-preserving** *iff, for all* $x,y\in P$ *we have that* $x\leq y$ *implies* $f(x)\leq f(y)$.



Definition. *A function from a finite ordered set* $(P, \leq)$ *to another ordered set* $(Q, \leq)$ *is called* **order-preserving** *iff, for all* $x, y \in P$ *we have that* $x \leq y$ *implies* $f(x) \leq f(y)$.

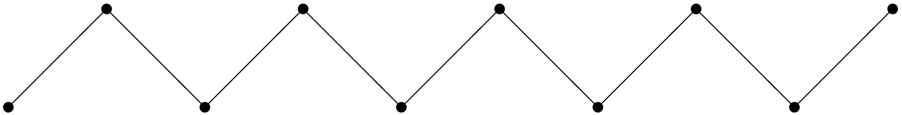


Definition. *A function from a finite ordered set* $(P,\leq)$ *to another ordered set* $(Q,\leq)$ *is called* **order-preserving** *iff, for all* $x,y\in P$ *we have that* $x\leq y$ *implies* $f(x)\leq f(y)$.



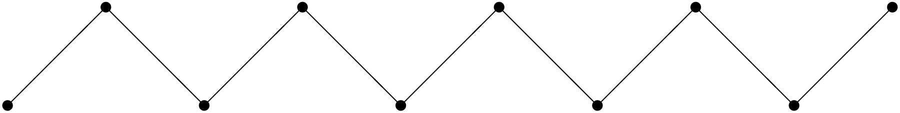
Definition. *A finite ordered set is said to have the* **fixed point property** *iff every order-preserving self-map has a fixed point.*

Definition. *A finite ordered set is said to have the* **fixed point property** *iff every order-preserving self-map has a fixed point.*

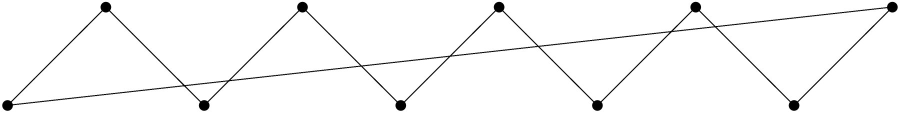


The Problem

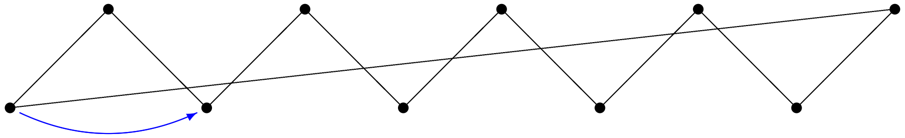
The Problem



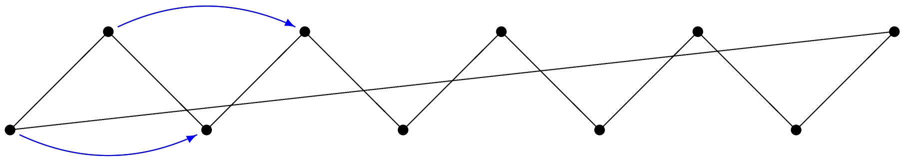
The Problem



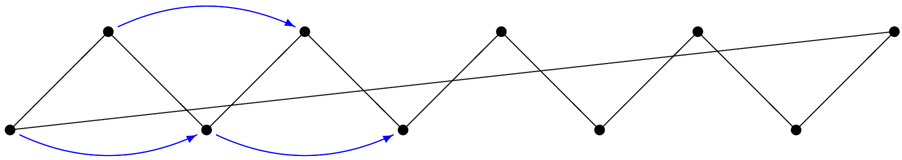
The Problem



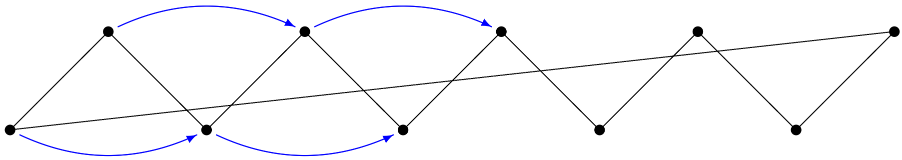
The Problem



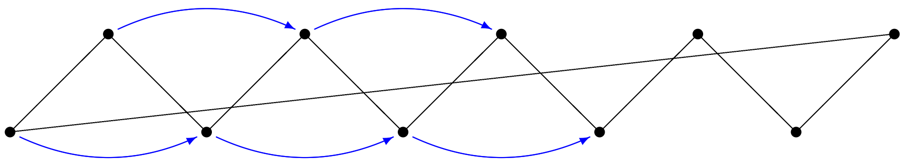
The Problem



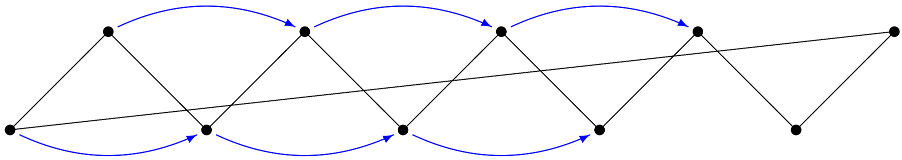
The Problem



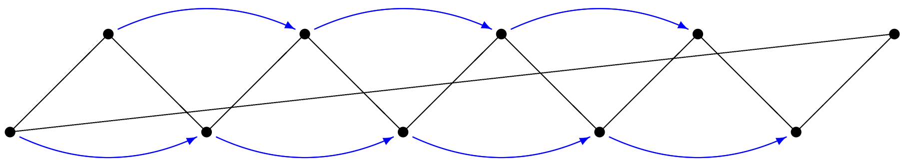
The Problem



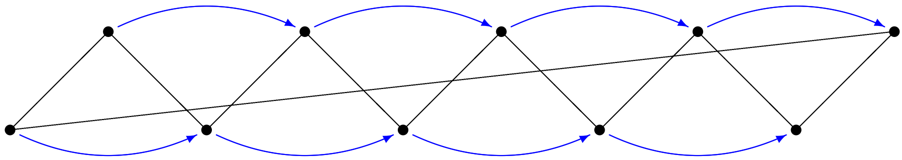
The Problem



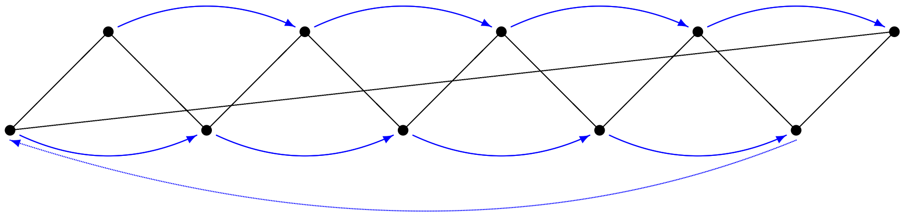
The Problem



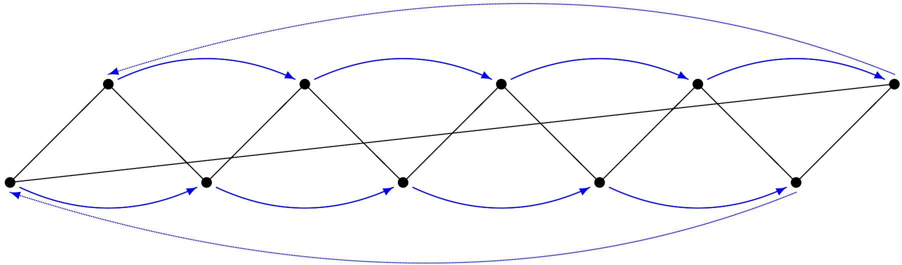
The Problem



The Problem



The Problem



Definition.

Definition. *Let P be an ordered set.*

Definition. *Let P be an ordered set. Then an order-preserving map $r : P \to P$ is called a* **retraction** *iff $r^2 = r$*

Definition. *Let P be an ordered set. Then an order-preserving map $r : P \to P$ is called a* **retraction** *iff $r^2 = r$ (that is, iff r is* **idempotent***).*

Definition. *Let P be an ordered set. Then an order-preserving map $r : P \to P$ is called a* **retraction** *iff $r^2 = r$ (that is, iff r is* **idempotent***). We will say that $R \subseteq P$ is a* **retract** *of P*

Definition. *Let P be an ordered set. Then an order-preserving map $r : P \to P$ is called a* **retraction** *iff $r^2 = r$ (that is, iff r is* **idempotent***). We will say that $R \subseteq P$ is a* **retract** *of P iff there is a retraction $r : P \to P$ with $r[P] = R$.*

Definition. *Let P be an ordered set. Then an order-preserving map $r : P \to P$ is called a* **retraction** *iff $r^2 = r$ (that is, iff r is* **idempotent***). We will say that $R \subseteq P$ is a* **retract** *of P iff there is a retraction $r : P \to P$ with $r[P] = R$.*

Theorem.

Definition. *Let P be an ordered set. Then an order-preserving map $r : P \to P$ is called a* **retraction** *iff $r^2 = r$ (that is, iff r is* **idempotent***). We will say that $R \subseteq P$ is a* **retract** *of P iff there is a retraction $r : P \to P$ with $r[P] = R$.*

Theorem. *Let P be an ordered set with the fixed point property and let $r : P \to P$ be a retraction.*

Definition. *Let P be an ordered set. Then an order-preserving map $r : P \to P$ is called a* **retraction** *iff $r^2 = r$ (that is, iff r is* **idempotent***). We will say that $R \subseteq P$ is a* **retract** *of P iff there is a retraction $r : P \to P$ with $r[P] = R$.*

Theorem. *Let P be an ordered set with the fixed point property and let $r : P \to P$ be a retraction. Then $r[P]$ has the fixed point property.*

Definition. *Let P be an ordered set. Then an order-preserving map $r : P \to P$ is called a* **retraction** *iff $r^2 = r$ (that is, iff r is* **idempotent***). We will say that $R \subseteq P$ is a* **retract** *of P iff there is a retraction $r : P \to P$ with $r[P] = R$.*

Theorem. *Let P be an ordered set with the fixed point property and let $r : P \to P$ be a retraction. Then $r[P]$ has the fixed point property.*

What about possible converses?

Definition.

Definition. *Let P be a finite ordered set and let $x, y \in P$.*

Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x*

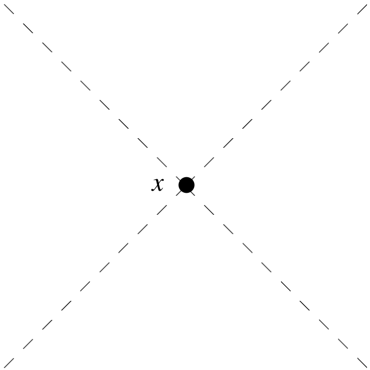
Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y.*

Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*

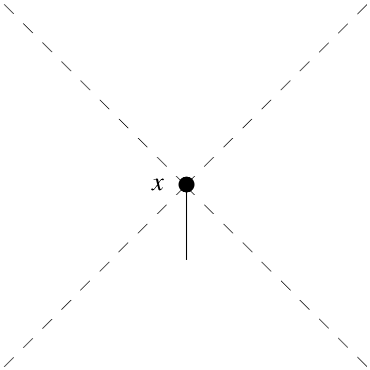
Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*

Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*

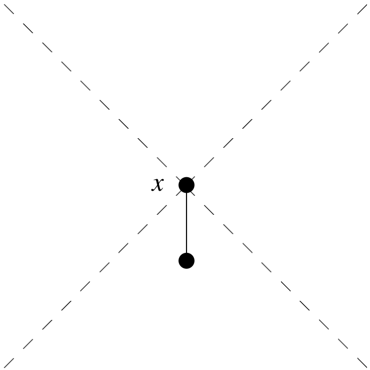
Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*



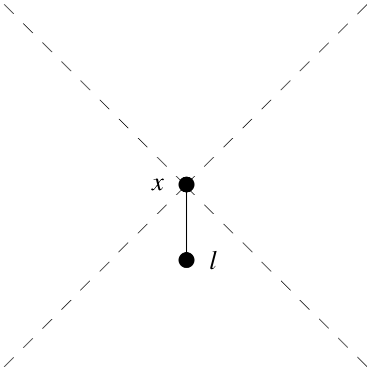
Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*



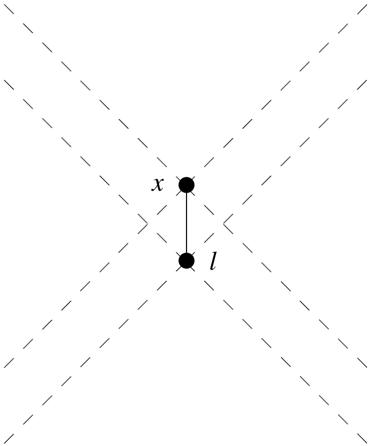
Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*



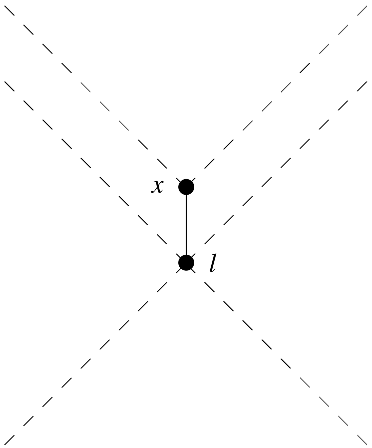
Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*



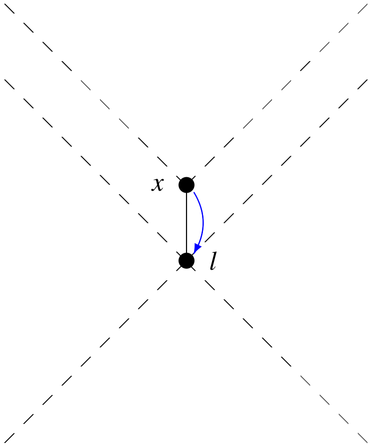
Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*



Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*



Definition. *Let P be a finite ordered set and let $x, y \in P$. If $x < y$ and there is no $z \in P$ so that $x < z < y$, then y is called an* **upper cover** *of x and x is called a* **lower cover** *of y. In a finite ordered set, a point is called* **irreducible** *iff it has exactly one upper cover or exactly one lower cover.*



Theorem (Rival, 1976).

Theorem (Rival, 1976). *Let P be a finite ordered set and let $a \in P$ be irreducible.*

Theorem (Rival, 1976). *Let P be a finite ordered set and let $a \in P$ be irreducible. Then P has the fixed point property iff*

Theorem (Rival, 1976). *Let P be a finite ordered set and let $a \in P$ be irreducible. Then P has the fixed point property iff $P \setminus \{a\}$ has the fixed point property.*

Theorem (Rival, 1976). *Let P be a finite ordered set and let $a \in P$ be irreducible. Then P has the fixed point property iff $P \setminus \{a\}$ has the fixed point property.*

Definition.

Theorem (Rival, 1976). *Let P be a finite ordered set and let $a \in P$ be irreducible. Then P has the fixed point property iff $P \setminus \{a\}$ has the fixed point property.*

Definition. *A finite ordered set P is called* **dismantlable** *iff*

Theorem (Rival, 1976). *Let P be a finite ordered set and let $a \in P$ be irreducible. Then P has the fixed point property iff $P \setminus \{a\}$ has the fixed point property.*

Definition. *A finite ordered set P is called* **dismantlable** *iff* $|P| = 1$

Theorem (Rival, 1976). *Let P be a finite ordered set and let $a \in P$ be irreducible. Then P has the fixed point property iff $P \setminus \{a\}$ has the fixed point property.*

Definition. *A finite ordered set P is called* **dismantlable** *iff $|P| = 1$ or there is a point $x \in P$ such that*

Theorem (Rival, 1976). *Let P be a finite ordered set and let $a \in P$ be irreducible. Then P has the fixed point property iff $P \setminus \{a\}$ has the fixed point property.*

Definition. *A finite ordered set P is called* **dismantlable** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. *x is irreducible in P,*

Theorem (Rival, 1976). *Let P be a finite ordered set and let $a \in P$ be irreducible. Then P has the fixed point property iff $P \setminus \{a\}$ has the fixed point property.*

Definition. *A finite ordered set P is called* **dismantlable** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. *x is irreducible in P,*
2. *$P \setminus \{x\}$ is dismantlable.*

Definition.

Definition. *Let P be a finite ordered set.*

Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point $b \in P \setminus \{a\}$)**

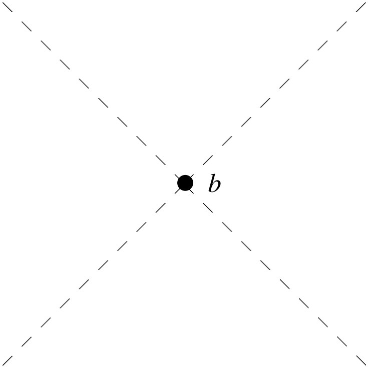
Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point** $b \in P \setminus \{a\}$**)** *iff for all $x \in P$, $x > a$ implies $x \geq b$*

Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point** $b \in P \setminus \{a\}$) *iff for all $x \in P$, $x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$.*

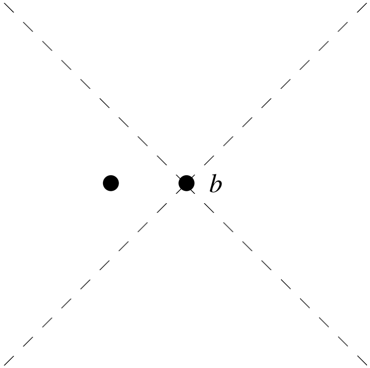
Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point** $b \in P \setminus \{a\}$**)** *iff for all $x \in P$, $x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$.*

Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point** $b \in P \setminus \{a\}$**)** *iff for all $x \in P$, $x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$.*

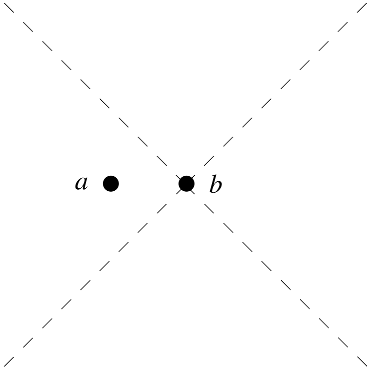
Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point** $b \in P \setminus \{a\}$**)** *iff for all $x \in P$, $x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$.*



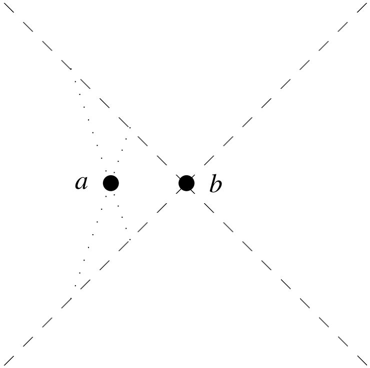
Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point** $b \in P \setminus \{a\}$**)** *iff for all $x \in P$, $x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$.*



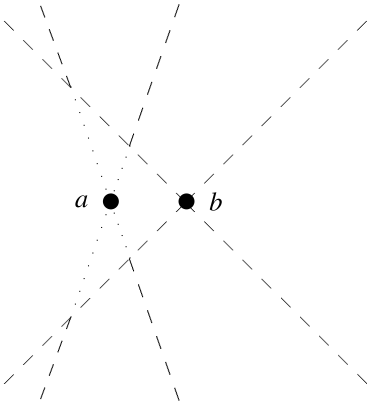
Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point $b \in P \setminus \{a\}$)** *iff for all $x \in P$, $x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$.*



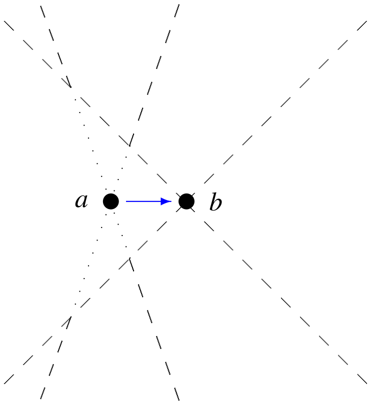
Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point** $b \in P \setminus \{a\}$**)** *iff for all $x \in P$, $x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$.*



Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point** $b \in P \setminus \{a\}$**)** *iff for all $x \in P$, $x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$.*



Definition. *Let P be a finite ordered set. A point $a \in P$ is called* **retractable (to the point** $b \in P \setminus \{a\}$) *iff for all $x \in P$, $x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$.*



Theorem (BS, 1993).

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable.*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition.

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff* $|P| = 1$

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. x *is retractable in P,*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. *$P \setminus \{a\}$ has the fixed point property.*
2. *$\updownarrow a \setminus \{a\}$ has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. *x is retractable in P,*
2. *$P \setminus \{x\}$ is connectedly collapsible,*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. x *is retractable in P,*
2. $P \setminus \{x\}$ *is connectedly collapsible,*
3. $(\updownarrow x) \setminus \{x\}$ *is connectedly collapsible.*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. x *is retractable in P,*
2. $P \setminus \{x\}$ *is connectedly collapsible,*
3. $(\updownarrow x) \setminus \{x\}$ *is connectedly collapsible.*

Definition.

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. x *is retractable in P,*
2. $P \setminus \{x\}$ *is connectedly collapsible,*
3. $(\updownarrow x) \setminus \{x\}$ *is connectedly collapsible.*

Definition. *A finite ordered set P is called* **collapsible** *iff*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. x *is retractable in P,*
2. $P \setminus \{x\}$ *is connectedly collapsible,*
3. $(\updownarrow x) \setminus \{x\}$ *is connectedly collapsible.*

Definition. *A finite ordered set P is called* **collapsible** *iff* $|P| \in \{0, 1\}$

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. *$P \setminus \{a\}$ has the fixed point property.*
2. *$\updownarrow a \setminus \{a\}$ has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. *x is retractable in P,*
2. *$P \setminus \{x\}$ is connectedly collapsible,*
3. *$(\updownarrow x) \setminus \{x\}$ is connectedly collapsible.*

Definition. *A finite ordered set P is called* **collapsible** *iff $|P| \in \{0, 1\}$ or there is a point $x \in P$ such that*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. x *is retractable in P,*
2. $P \setminus \{x\}$ *is connectedly collapsible,*
3. $(\updownarrow x) \setminus \{x\}$ *is connectedly collapsible.*

Definition. *A finite ordered set P is called* **collapsible** *iff $|P| \in \{0, 1\}$ or there is a point $x \in P$ such that*

1. x *is retractable in P,*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

1. *x is retractable in P,*
2. *$P \setminus \{x\}$ is connectedly collapsible,*
3. *$(\updownarrow x) \setminus \{x\}$ is connectedly collapsible.*

Definition. *A finite ordered set P is called* **collapsible** *iff $|P| \in \{0, 1\}$ or there is a point $x \in P$ such that*

1. *x is retractable in P,*
2. *$P \setminus \{x\}$ is collapsible,*

Theorem (BS, 1993). *Let P be a finite ordered set and let $a \in P$ be retractable. Then P has the fixed point property iff*

1. $P \setminus \{a\}$ *has the fixed point property.*
2. $\updownarrow a \setminus \{a\}$ *has the fixed point property.*

Definition. *A finite ordered set P is called* **connectedly collapsible** *iff $|P| = 1$ or there is a point $x \in P$ such that*

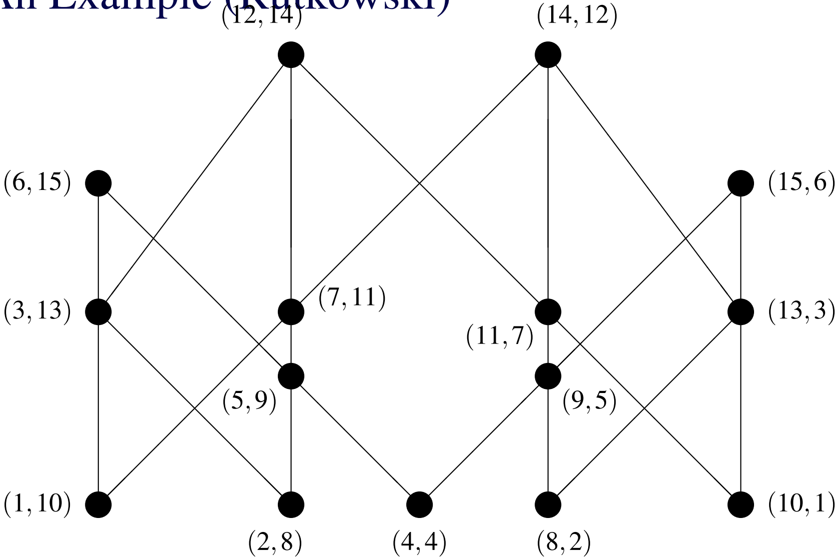
1. x *is retractable in P,*
2. $P \setminus \{x\}$ *is connectedly collapsible,*
3. $(\updownarrow x) \setminus \{x\}$ *is connectedly collapsible.*

Definition. *A finite ordered set P is called* **collapsible** *iff $|P| \in \{0, 1\}$ or there is a point $x \in P$ such that*

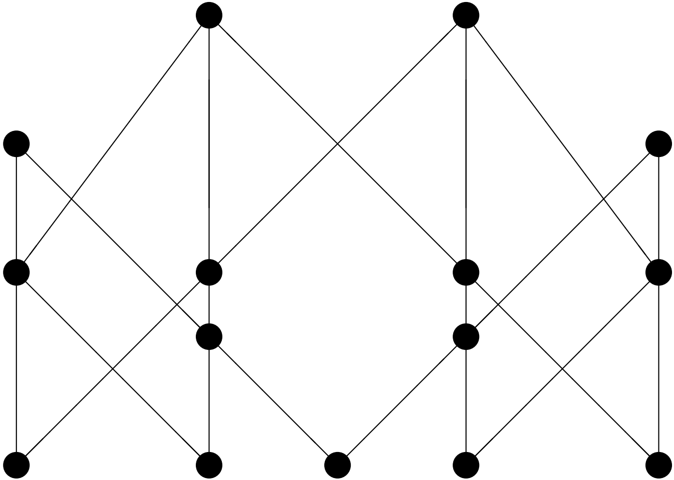
1. x *is retractable in P,*
2. $P \setminus \{x\}$ *is collapsible,*
3. $(\updownarrow x) \setminus \{x\}$ *is collapsible.*

An Example (Rutkowski)

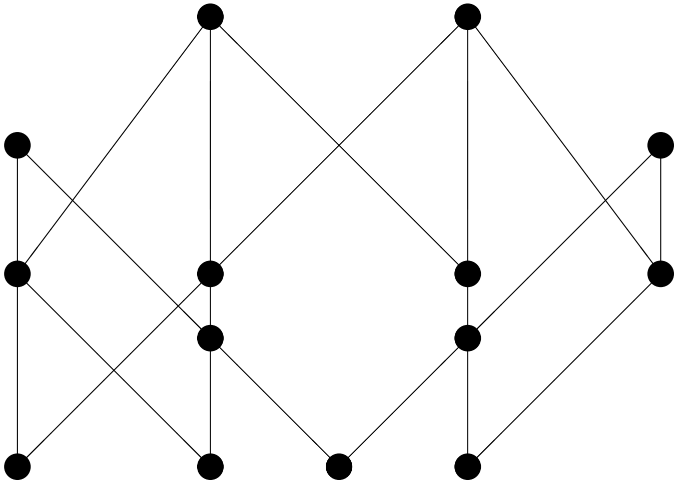
An Example (Rutkowski)



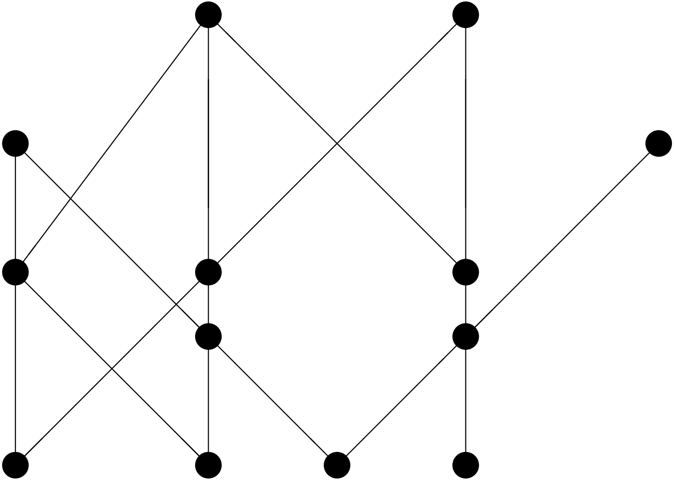
An Example (Rutkowski)



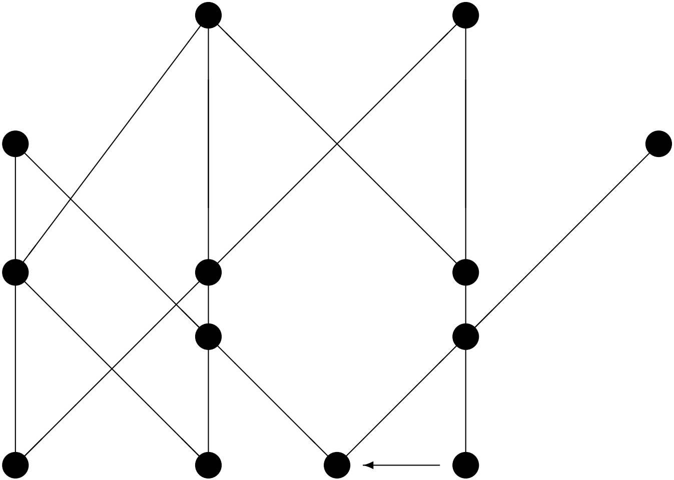
An Example (Rutkowski)



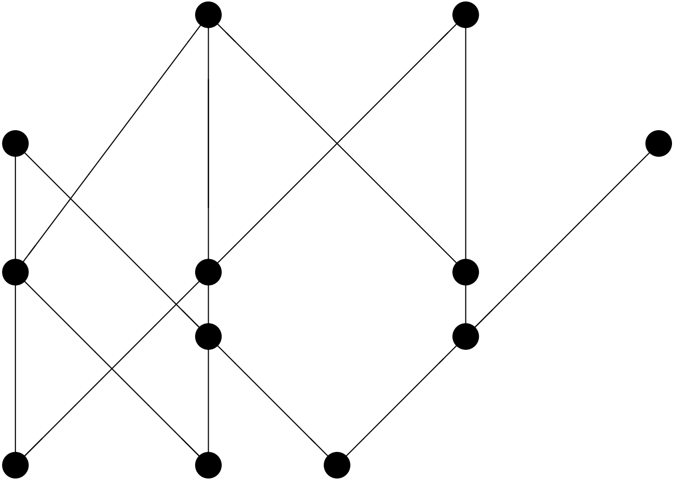
An Example (Rutkowski)



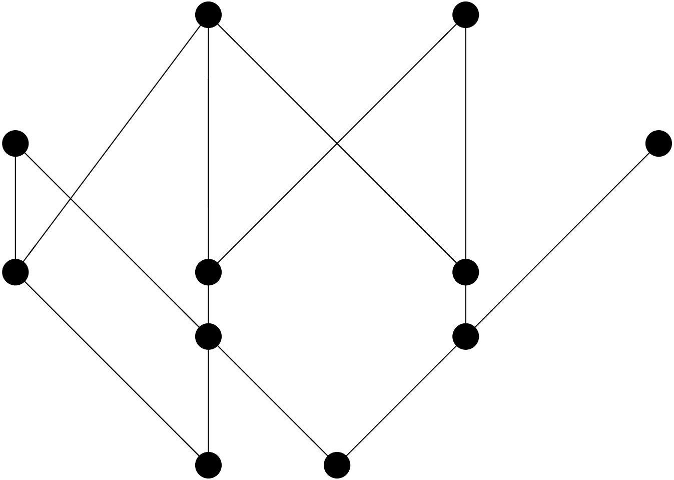
An Example (Rutkowski)



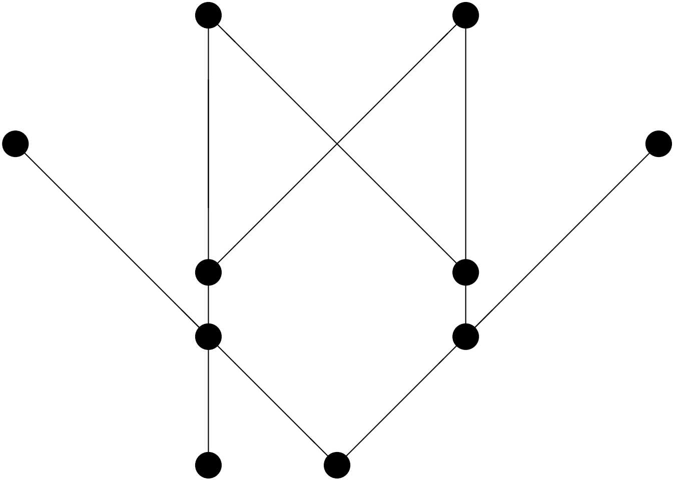
An Example (Rutkowski)



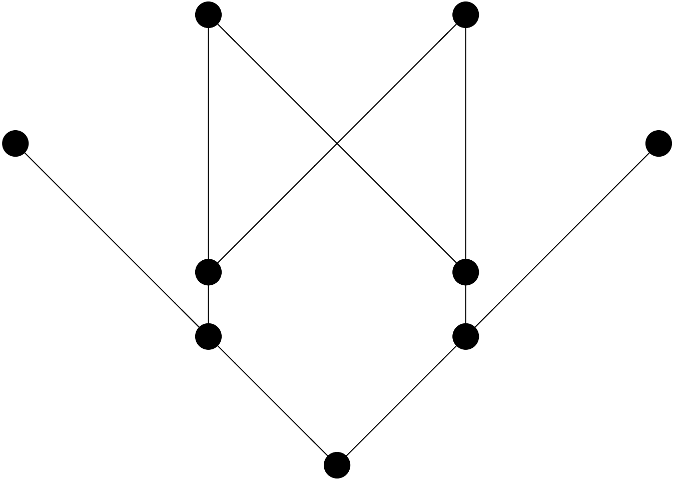
An Example (Rutkowski)



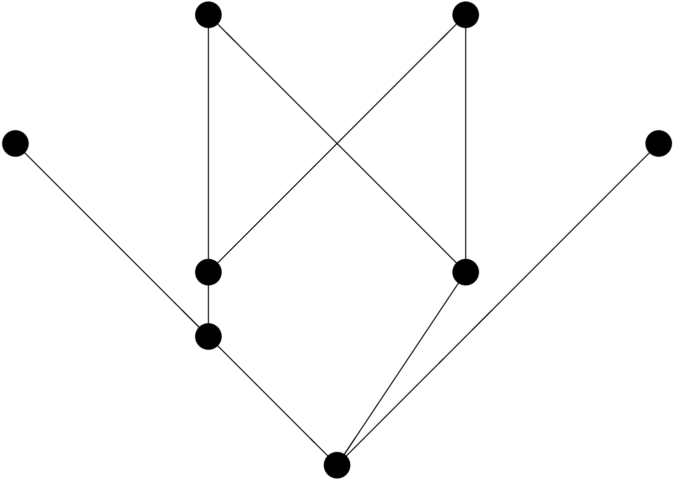
An Example (Rutkowski)



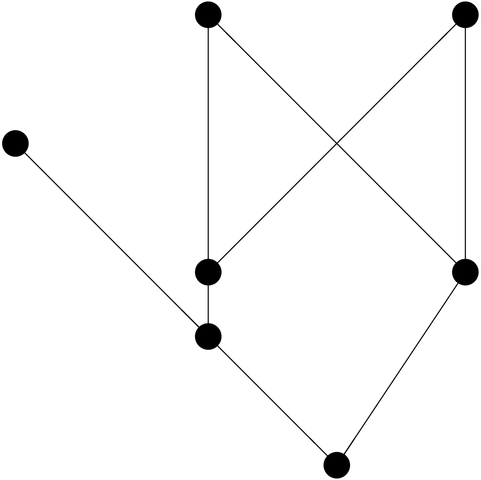
An Example (Rutkowski)



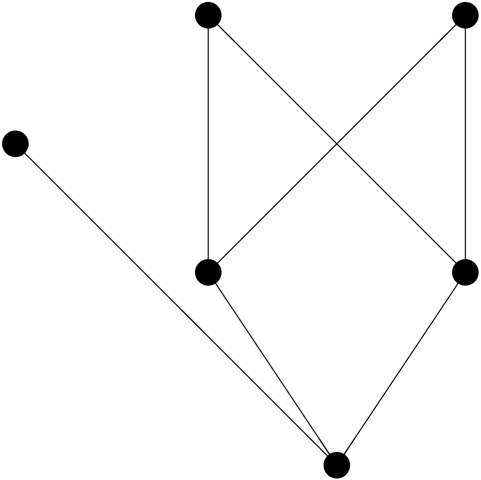
An Example (Rutkowski)



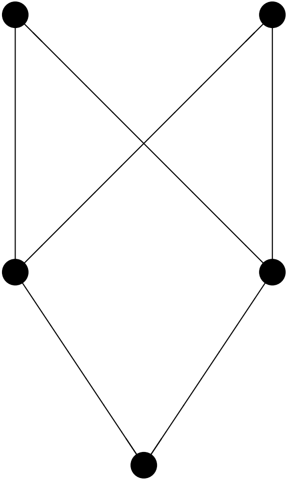
An Example (Rutkowski)



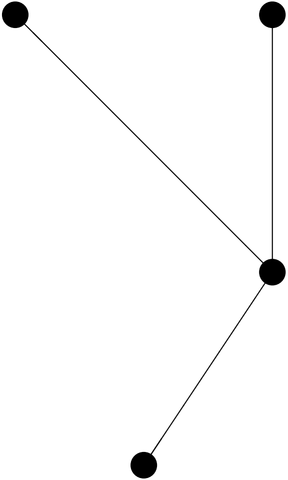
An Example (Rutkowski)



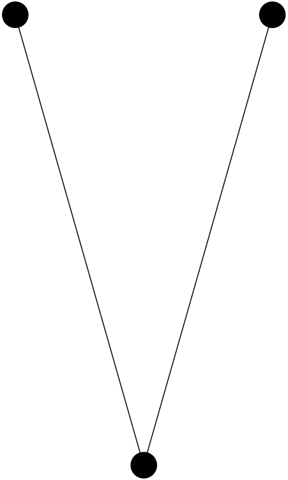
An Example (Rutkowski)



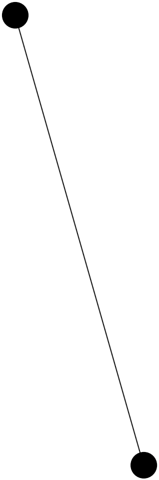
An Example (Rutkowski)



An Example (Rutkowski)



An Example (Rutkowski)



An Example (Rutkowski)

Theorem (Fofanova, Rutkowski, Rival, 1994).

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2.*

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or*

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or, P contains a minimal element a that is retractable to another minimal element b.*

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or, P contains a minimal element a that is retractable to another minimal element b.*

Definition.

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or, P contains a minimal element a that is retractable to another minimal element b.*

Definition. *Let P be a finite ordered set and let $a \in P$. Then a is called* **d2-retractable** *(in P) iff one of the following holds.*

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or, P contains a minimal element a that is retractable to another minimal element b.*

Definition. *Let P be a finite ordered set and let $a \in P$. Then a is called* **d2-retractable** *(in P) iff one of the following holds.*

1. *The point a is irreducible, or,*

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or, P contains a minimal element a that is retractable to another minimal element b.*

Definition. *Let P be a finite ordered set and let $a \in P$. Then a is called* **d2-retractable** *(in P) iff one of the following holds.*

1. *The point a is irreducible, or,*
2. *The point a is not irreducible, but minimal in P and retractable to another minimal element b of P.*

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or, P contains a minimal element a that is retractable to another minimal element b.*

Definition. *Let P be a finite ordered set and let $a \in P$. Then a is called* **d2-retractable** *(in P) iff one of the following holds.*

1. *The point a is irreducible, or,*
2. *The point a is not irreducible, but minimal in P and retractable to another minimal element b of P.*

Definition.

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or, P contains a minimal element a that is retractable to another minimal element b.*

Definition. *Let P be a finite ordered set and let $a \in P$. Then a is called* **d2-retractable** *(in P) iff one of the following holds.*

1. *The point a is irreducible, or,*
2. *The point a is not irreducible, but minimal in P and retractable to another minimal element b of P.*

Definition. *Let P be a finite ordered set.*

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or, P contains a minimal element a that is retractable to another minimal element b.*

Definition. *Let P be a finite ordered set and let $a \in P$. Then a is called* **d2-retractable** *(in P) iff one of the following holds.*

1. *The point a is irreducible, or,*
2. *The point a is not irreducible, but minimal in P and retractable to another minimal element b of P.*

Definition. *Let P be a finite ordered set. Then P is said to have a* **d2-retraction sequence**

Theorem (Fofanova, Rutkowski, Rival, 1994). *Let P be a finite ordered set of interval dimension 2. Then P contains a point with a unique lower cover that is minimal in P, or, P contains a minimal element a that is retractable to another minimal element b.*

Definition. *Let P be a finite ordered set and let $a \in P$. Then a is called* **d2-retractable** *(in P) iff one of the following holds.*

1. *The point a is irreducible, or,*
2. *The point a is not irreducible, but minimal in P and retractable to another minimal element b of P.*

Definition. *Let P be a finite ordered set. Then P is said to have a* **d2-retraction sequence** *iff there is a retraction sequence $a_1, \ldots, a_{|P|}$ so that, for all $i \in \{1, \ldots, |P|-1\}$, the point a_i is d2-retractable in $P \setminus \{a_1, \ldots, a_{i-1}\}$.*

Lemma (BS, henceforth omitted, joke gets old).

Lemma (BS, henceforth omitted, joke gets old). *Let P be a finite ordered set that has a d2-retraction sequence and let $r : P \to P$ be a retraction.*

Lemma (BS, henceforth omitted, joke gets old). *Let P be a finite ordered set that has a d2-retraction sequence and let $r : P \to P$ be a retraction. Then $r[P]$ has a d2-retraction sequence.*

Lemma (BS, henceforth omitted, joke gets old). *Let P be a finite ordered set that has a d2-retraction sequence and let $r : P \to P$ be a retraction. Then $r[P]$ has a d2-retraction sequence.*

Lemma.

Lemma (BS, henceforth omitted, joke gets old). *Let P be a finite ordered set that has a d2-retraction sequence and let $r : P \to P$ be a retraction. Then $r[P]$ has a d2-retraction sequence.*

Lemma. *Let P be a finite ordered set and let $x \in P$ be d2-retractable.*

Lemma (BS, henceforth omitted, joke gets old). *Let P be a finite ordered set that has a d2-retraction sequence and let $r : P \to P$ be a retraction. Then $r[P]$ has a d2-retraction sequence.*

Lemma. *Let P be a finite ordered set and let $x \in P$ be d2-retractable. Then P has a d2-retraction sequence iff $P \setminus \{x\}$ does.*

Lemma (BS, henceforth omitted, joke gets old). *Let P be a finite ordered set that has a d2-retraction sequence and let $r : P \to P$ be a retraction. Then $r[P]$ has a d2-retraction sequence.*

Lemma. *Let P be a finite ordered set and let $x \in P$ be d2-retractable. Then P has a d2-retraction sequence iff $P \setminus \{x\}$ does.*

Lemma.

Lemma (BS, henceforth omitted, joke gets old). *Let P be a finite ordered set that has a d2-retraction sequence and let $r : P \to P$ be a retraction. Then $r[P]$ has a d2-retraction sequence.*

Lemma. *Let P be a finite ordered set and let $x \in P$ be d2-retractable. Then P has a d2-retraction sequence iff $P \setminus \{x\}$ does.*

Lemma. *For any ordered set P, it takes at most $3|P|^3$ steps to decide if there is a d2-retraction sequence for P*

Lemma (BS, henceforth omitted, joke gets old). *Let P be a finite ordered set that has a d2-retraction sequence and let $r : P \to P$ be a retraction. Then $r[P]$ has a d2-retraction sequence.*

Lemma. *Let P be a finite ordered set and let $x \in P$ be d2-retractable. Then P has a d2-retraction sequence iff $P \setminus \{x\}$ does.*

Lemma. *For any ordered set P, it takes at most $3|P|^3$ steps to decide if there is a d2-retraction sequence for P and, if so, to compute a d2-retraction sequence for P.*

Lemma.

Lemma. *Let P be a finite ordered set that has a d2-retraction sequence $a_1, \ldots, a_{|P|}$.*

Lemma. *Let P be a finite ordered set that has a d2-retraction sequence $a_1, \ldots, a_{|P|}$. Then P has the fixed point property iff*

Lemma. *Let P be a finite ordered set that has a d2-retraction sequence $a_1, \ldots, a_{|P|}$. Then P has the fixed point property iff, for all $i \in \{1, \ldots, |P|-1\}$ so that, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, a_i is minimal, not irreducible, but retractable to minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, we have that $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i$ has the fixed point property.*

Lemma.

Lemma. *Let P be a finite ordered set.*

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton*

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton, or, $|P| > 1$ and P has a d2-retraction sequence $a_1, \ldots, a_{|P|}$ so that*

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton, or, $|P| > 1$ and P has a d2-retraction sequence $a_1, \ldots, a_{|P|}$ so that, for each $i \in \{1, \ldots, |P| - 1\}$, if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$*

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton, or, $|P|>1$ and P has a d2-retraction sequence $a_1,\ldots,a_{|P|}$ so that, for each $i\in\{1,\ldots,|P|-1\}$, if, in $P\setminus\{a_1,\ldots,a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P\setminus\{a_1,\ldots,a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i\setminus\{a_i\}$ is d2-collapsible.*

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton, or, $|P| > 1$ and P has a d2-retraction sequence $a_1, \ldots, a_{|P|}$ so that, for each $i \in \{1, \ldots, |P| - 1\}$, if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible.*

d2-collapsible*: Don't allow the empty set.*

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton, or, $|P| > 1$ and P has a d2-retraction sequence $a_1, \ldots, a_{|P|}$ so that, for each $i \in \{1, \ldots, |P|-1\}$, if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible.*

d2-collapsible*: Don't allow the empty set.*

Proposition.

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton, or, $|P| > 1$ and P has a d2-retraction sequence $a_1, \ldots, a_{|P|}$ so that, for each $i \in \{1, \ldots, |P|-1\}$, if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible.*
d2-collapsible*: Don't allow the empty set.*

Proposition. *Let P be a finite ordered set.*

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton, or, $|P| > 1$ and P has a d2-retraction sequence $a_1, \ldots, a_{|P|}$ so that, for each $i \in \{1, \ldots, |P|-1\}$, if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible.*

d2-collapsible*: Don't allow the empty set.*

Proposition. *Let P be a finite ordered set.*

1. *If P is d2-connectedly collapsible, then P is d2-collapsible.*

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton, or, $|P| > 1$ and P has a d2-retraction sequence $a_1, \ldots, a_{|P|}$ so that, for each $i \in \{1, \ldots, |P|-1\}$, if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible.*

d2-collapsible*: Don't allow the empty set.*

Proposition. *Let P be a finite ordered set.*

1. *If P is d2-connectedly collapsible, then P is d2-collapsible.*
2. *If P is d2-connectedly collapsible, then P has the fixed point property.*

Lemma. *Let P be a finite ordered set. Then P is called* **d2-collapsible** *iff P is empty, a singleton, or, $|P| > 1$ and P has a d2-retraction sequence $a_1, \ldots, a_{|P|}$ so that, for each $i \in \{1, \ldots, |P| - 1\}$, if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P \setminus \{a_1, \ldots, a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible.*

d2-collapsible*: Don't allow the empty set.*

Proposition. *Let P be a finite ordered set.*

1. *If P is d2-connectedly collapsible, then P is d2-collapsible.*
2. *If P is d2-connectedly collapsible, then P has the fixed point property.*
3. *If P is d2-collapsible, then P has the fixed point property iff P is d2-connectedly collapsible.*

Lemma.

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction.*

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible.*

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible. Moreover, if P is d2-connectedly collapsible, then $r[P]$ is d2-connectedly collapsible.*

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible. Moreover, if P is d2-connectedly collapsible, then $r[P]$ is d2-connectedly collapsible.*

Lemma.

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible. Moreover, if P is d2-connectedly collapsible, then $r[P]$ is d2-connectedly collapsible.*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be <u>any</u> d2-retraction sequence for P.*

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible. Moreover, if P is d2-connectedly collapsible, then $r[P]$ is d2-connectedly collapsible.*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be <u>any</u> d2-retraction sequence for P. Then P is d2-collapsible iff*

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible. Moreover, if P is d2-connectedly collapsible, then $r[P]$ is d2-connectedly collapsible.*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be* <u>any</u> *d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P| - 1\}$, we have that*

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible. Moreover, if P is d2-connectedly collapsible, then $r[P]$ is d2-connectedly collapsible.*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be <u>any</u> d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P|-1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$*

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible. Moreover, if P is d2-connectedly collapsible, then $r[P]$ is d2-connectedly collapsible.*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be any d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P|-1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible.*

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible. Moreover, if P is d2-connectedly collapsible, then $r[P]$ is d2-connectedly collapsible.*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be any d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P| - 1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible. The same result holds when "d2-collapsible" is replaced with "d2-connectedly collapsible."*

Lemma. *Let P be a d2-collapsible ordered set and let $r : P \to P$ be a retraction. Then $r[P]$ is d2-collapsible. Moreover, if P is d2-connectedly collapsible, then $r[P]$ is d2-connectedly collapsible.*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be any d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P|-1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible. The same result holds when "d2-collapsible" is replaced with "d2-connectedly collapsible."*

Problem: Recursively, that's a lot of sets.

Definition.

Definition. *Let C be a finite ordered set.*

Definition. *Let C be a finite ordered set. Then C is called a* **core** *iff P contains no irreducible elements.*

Definition. *Let C be a finite ordered set. Then C is called a* **core** *iff P contains no irreducible elements.*

Theorem.

Definition. *Let C be a finite ordered set. Then C is called a* **core** *iff P contains no irreducible elements.*

Theorem. *(Duffus-Poguntke-Rival 1980, Farley 1993) Every finite ordered set has a unique core.*

Definition. *Let C be a finite ordered set. Then C is called a* **core** *iff P contains no irreducible elements.*

Theorem. *(Duffus-Poguntke-Rival 1980, Farley 1993) Every finite ordered set has a unique core.*

Lemma.

Definition. *Let C be a finite ordered set. Then C is called a* **core** *iff P contains no irreducible elements.*

Theorem. *(Duffus-Poguntke-Rival 1980, Farley 1993) Every finite ordered set has a unique core.*

Lemma. *Let P be a finite ordered set. Then P is d2-collapsible iff its core is d2-collapsible.*

Definition. *Let C be a finite ordered set. Then C is called a* **core** *iff P contains no irreducible elements.*

Theorem. *(Duffus-Poguntke-Rival 1980, Farley 1993) Every finite ordered set has a unique core.*

Lemma. *Let P be a finite ordered set. Then P is d2-collapsible iff its core is d2-collapsible. The same result holds for d2-connectedly collapsible ordered sets.*

Lemma.

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P.*

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Definition.

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Definition. *Let P be a finite ordered and let $Q \subseteq P$.*

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Definition. *Let P be a finite ordered and let $Q \subseteq P$. Then P has a* **retraction sequence to** *Q iff*

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Definition. *Let P be a finite ordered and let $Q \subseteq P$. Then P has a* **retraction sequence to** *Q iff the elements of $P \setminus Q$ can be arranged in a sequence $a_1, \ldots, a_{|P|-|Q|}$ so that, for all $i \in \{1, \ldots, |P|-1\}$, the point a_i is d2-retractable in $P \setminus \{a_1, \ldots, a_{i-1}\}$.*

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Definition. *Let P be a finite ordered and let $Q \subseteq P$. Then P has a* **retraction sequence to** *Q iff the elements of $P \setminus Q$ can be arranged in a sequence $a_1, \ldots, a_{|P|-|Q|}$ so that, for all $i \in \{1, \ldots, |P|-1\}$, the point a_i is d2-retractable in $P \setminus \{a_1, \ldots, a_{i-1}\}$.*

Lemma.

Lemma. *Let H be a finite ordered set, let $x,y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Definition. *Let P be a finite ordered and let $Q \subseteq P$. Then P has a* **retraction sequence to** *Q iff the elements of $P \setminus Q$ can be arranged in a sequence $a_1, \ldots, a_{|P|-|Q|}$ so that, for all $i \in \{1, \ldots, |P|-1\}$, the point a_i is d2-retractable in $P \setminus \{a_1, \ldots, a_{i-1}\}$.*

Lemma. *Let P be a finite ordered set, let $x,y \in P$ be two incomparable elements*

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Definition. *Let P be a finite ordered and let $Q \subseteq P$. Then P has a* **retraction sequence to** *Q iff the elements of $P \setminus Q$ can be arranged in a sequence $a_1, \ldots, a_{|P|-|Q|}$ so that, for all $i \in \{1, \ldots, |P|-1\}$, the point a_i is d2-retractable in $P \setminus \{a_1, \ldots, a_{i-1}\}$.*

Lemma. *Let P be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $Q \subseteq P$ be a subset of P that contains x and y*

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Definition. *Let P be a finite ordered and let $Q \subseteq P$. Then P has a* **retraction sequence to** *Q iff the elements of $P \setminus Q$ can be arranged in a sequence $a_1, \ldots, a_{|P|-|Q|}$ so that, for all $i \in \{1, \ldots, |P|-1\}$, the point a_i is d2-retractable in $P \setminus \{a_1, \ldots, a_{i-1}\}$.*

Lemma. *Let P be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $Q \subseteq P$ be a subset of P that contains x and y and for which P has a d2-retraction sequence to Q.*

Lemma. *Let H be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $a \in \uparrow x \cap \uparrow y$ be irreducible in P. Then a is irreducible in $\uparrow x \cap \uparrow y$.*

Definition. *Let P be a finite ordered and let $Q \subseteq P$. Then P has a* **retraction sequence to** *Q iff the elements of $P \setminus Q$ can be arranged in a sequence $a_1, \ldots, a_{|P|-|Q|}$ so that, for all $i \in \{1, \ldots, |P|-1\}$, the point a_i is d2-retractable in $P \setminus \{a_1, \ldots, a_{i-1}\}$.*

Lemma. *Let P be a finite ordered set, let $x, y \in P$ be two incomparable elements and let $Q \subseteq P$ be a subset of P that contains x and y and for which P has a d2-retraction sequence to Q. Then the core of $\uparrow_Q x \cap \uparrow_Q y$ is isomorphic to the core of $\uparrow x \cap \uparrow y$.*

Lemma.

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be <u>any</u> d2-retraction sequence for P.*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be* <u>*any*</u> *d2-retraction sequence for P. Then P is d2-collapsible iff*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be any d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P| - 1\}$, we have that*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be any d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P|-1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element a_{i+j} of $P \setminus \{a_1, \ldots, a_{i-1}\}$*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be* <u>*any*</u> *d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P|-1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element a_{i+j} of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P \setminus \{a_1, \ldots, a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible.*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be <u>any</u> d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P|-1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element a_{i+j} of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P \setminus \{a_1, \ldots, a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible. The same result holds when "d2-collapsible" is replaced with "d2-connectedly collapsible."*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be any d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P|-1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element a_{i+j} of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P \setminus \{a_1, \ldots, a_{i-1}\}} a_i \setminus \{a_i\}$ is d2-collapsible. The same result holds when "d2-collapsible" is replaced with "d2-connectedly collapsible."*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be any d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P|-1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element a_{i+j} of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_i \cap \uparrow_{P\setminus\{a_1,\ldots,a_{i-1}\}} a_{i+j}$ is d2-collapsible. The same result holds when "d2-collapsible" is replaced with "d2-connectedly collapsible."*

Lemma. *Let P be a finite ordered set with a d2-retraction sequence and let $a_1, \ldots, a_{|P|}$ be* <u>*any*</u> *d2-retraction sequence for P. Then P is d2-collapsible iff, for any $i \in \{1, \ldots, |P|-1\}$, we have that if, in $P \setminus \{a_1, \ldots, a_{i-1}\}$, the point a_i is minimal, not irreducible, but retractable to another minimal element a_{i+j} of $P \setminus \{a_1, \ldots, a_{i-1}\}$, then $\uparrow a_i \cap \uparrow a_{i+j}$ is d2-collapsible. The same result holds when "d2-collapsible" is replaced with "d2-connectedly collapsible."*

Algorithm.

Algorithm. *The algorithm* **D2CCcheck**.

Algorithm. *The algorithm* **D2CCcheck**.

1. *Input: The algorithm takes as input a nonempty list* $(P_1, \ldots, P_n)$ *of ordered sets.*

Algorithm. *The algorithm* **D2CCcheck**.

1. *Input: The algorithm takes as input a nonempty list* $(P_1, \ldots, P_n)$ *of ordered sets.*
2. *Initialization.*

Algorithm. *The algorithm* **D2CCcheck**.

1. *Input: The algorithm takes as input a nonempty list* $(P_1, \ldots, P_n)$ *of ordered sets.*
2. *Initialization.*
   2.1 *Initialize the list* $\mathscr{S}$ *to be the input,* $\mathscr{S} := (P_1, \ldots, P_n)$.

Algorithm. *The algorithm* **D2CCcheck**.

1. *Input: The algorithm takes as input a nonempty list* $(P_1, \ldots, P_n)$ *of ordered sets.*
2. *Initialization.*
    2.1 *Initialize the list* $\mathscr{S}$ *to be the input,* $\mathscr{S} := (P_1, \ldots, P_n)$.
    2.2 *Initialize the list* $\mathscr{D}$ *to be the empty list.*

Algorithm. *The algorithm* **D2CCcheck**.

1. *Input: The algorithm takes as input a nonempty list $(P_1, \ldots, P_n)$ of ordered sets.*
2. *Initialization.*
   2.1 *Initialize the list $\mathscr{S}$ to be the input, $\mathscr{S} := (P_1, \ldots, P_n)$.*
   2.2 *Initialize the list $\mathscr{D}$ to be the empty list.*
   2.3 *Set the counter $c := 1$.*

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$*, return* $\mathscr{S}$*,* $\mathscr{D}$*, exit.*

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$*, return* $\mathscr{S}$*,* $\mathscr{D}$*, exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$*, consider the set* Q_c*.*

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$, *return* $\mathscr{S}$, $\mathscr{D}$, *exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$, *consider the set* Q_c.
   4.1 *If* Q_c *is a singleton, append* $d_c :=$ *TRUE to* $\mathscr{D}$.

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$*, return* $\mathscr{S}$, $\mathscr{D}$*, exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$*, consider the set* Q_c*.*
   4.1 *If* Q_c *is a singleton, append* $d_c :=$ *TRUE to* $\mathscr{D}$*.*
   4.2 *If* Q_c *is an antichain and* $|Q_c| > 1$*, append* $d_c :=$ *AC to* $\mathscr{D}$*.*

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$*, return* $\mathscr{S}$*,* $\mathscr{D}$*, exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$*, consider the set* Q_c*.*
   - 4.1 *If* Q_c *is a singleton, append* $d_c :=$ *TRUE to* $\mathscr{D}$*.*
   - 4.2 *If* Q_c *is an antichain and* $|Q_c| > 1$*, append* $d_c :=$ *AC to* $\mathscr{D}$*.*
   - 4.3 *If* Q_c *is not an antichain and* $|Q_c| > 1$*, execute an algorithm to find a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$ *for* Q_c*.*

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$*, return* $\mathscr{S}$, $\mathscr{D}$*, exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$*, consider the set* Q_c*.*
   - 4.1 *If* Q_c *is a singleton, append* $d_c :=$ *TRUE to* $\mathscr{D}$*.*
   - 4.2 *If* Q_c *is an antichain and* $|Q_c| > 1$*, append* $d_c :=$ *AC to* $\mathscr{D}$*.*
   - 4.3 *If* Q_c *is not an antichain and* $|Q_c| > 1$*, execute an algorithm to find a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$ *for* Q_c*.*
     - 4.3.1 *If* Q_c *does not have a d2-retraction sequence, append* $d_c :=$ *NO to* $\mathscr{D}$*.*

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$*, return* $\mathscr{S}$*,* $\mathscr{D}$*, exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$*, consider the set* Q_c*.*
   - 4.1 *If* Q_c *is a singleton, append* $d_c :=$ *TRUE to* $\mathscr{D}$*.*
   - 4.2 *If* Q_c *is an antichain and* $|Q_c| > 1$*, append* $d_c :=$ *AC to* $\mathscr{D}$*.*
   - 4.3 *If* Q_c *is not an antichain and* $|Q_c| > 1$*, execute an algorithm to find a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$ *for* Q_c*.*
     - 4.3.1 *If* Q_c *does not have a d2-retraction sequence, append* $d_c :=$ *NO to* $\mathscr{D}$*.*
     - 4.3.2 *If* Q_c *has a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$*, append* $d_c :=$ *TRUE to* $\mathscr{D}$ *and proceed as follows.*

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$*, return* $\mathscr{S}$*,* $\mathscr{D}$*, exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$*, consider the set* Q_c*.*
   - 4.1 *If* Q_c *is a singleton, append* $d_c :=$ *TRUE to* $\mathscr{D}$*.*
   - 4.2 *If* Q_c *is an antichain and* $|Q_c| > 1$*, append* $d_c :=$ *AC to* $\mathscr{D}$*.*
   - 4.3 *If* Q_c *is not an antichain and* $|Q_c| > 1$*, execute an algorithm to find a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$ *for* Q_c*.*
     - 4.3.1 *If* Q_c *does not have a d2-retraction sequence, append* $d_c :=$ *NO to* $\mathscr{D}$*.*
     - 4.3.2 *If* Q_c *has a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$*, append* $d_c :=$ *TRUE to* $\mathscr{D}$ *and proceed as follows. For every* $i \in \{1, \ldots, |Q_c| - 1\}$ *so that, in* $Q_c \setminus \{a_1, \ldots, a_{i-1}\}$*,* a_i *is a minimal element that is not irreducible, but retractable to the minimal element* a_{i+j} *of* $Q_c \setminus \{a_1, \ldots, a_{i-1}\}$

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$, *return* $\mathscr{S}$, $\mathscr{D}$, *exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$, *consider the set* Q_c.
   - 4.1 *If* Q_c *is a singleton, append* $d_c :=$ *TRUE to* $\mathscr{D}$.
   - 4.2 *If* Q_c *is an antichain and* $|Q_c| > 1$, *append* $d_c :=$ *AC to* $\mathscr{D}$.
   - 4.3 *If* Q_c *is not an antichain and* $|Q_c| > 1$, *execute an algorithm to find a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$ *for* Q_c.
     - 4.3.1 *If* Q_c *does not have a d2-retraction sequence, append* $d_c :=$ *NO to* $\mathscr{D}$.
     - 4.3.2 *If* Q_c *has a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$, *append* $d_c :=$ *TRUE to* $\mathscr{D}$ *and proceed as follows. For every* $i \in \{1, \ldots, |Q_c| - 1\}$ *so that, in* $Q_c \setminus \{a_1, \ldots, a_{i-1}\}$, a_i *is a minimal element that is not irreducible, but retractable to the minimal element* a_{i+j} *of* $Q_c \setminus \{a_1, \ldots, a_{i-1}\}$, *check if* $\uparrow a_i \cap \uparrow a_{i+j}$ *is already listed in* $\mathscr{S}$.

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$*, return* $\mathscr{S}$, $\mathscr{D}$*, exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$*, consider the set* Q_c*.*
   - 4.1 *If* Q_c *is a singleton, append* $d_c :=$ *TRUE to* $\mathscr{D}$*.*
   - 4.2 *If* Q_c *is an antichain and* $|Q_c| > 1$*, append* $d_c :=$ *AC to* $\mathscr{D}$*.*
   - 4.3 *If* Q_c *is not an antichain and* $|Q_c| > 1$*, execute an algorithm to find a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$ *for* Q_c*.*
     - 4.3.1 *If* Q_c *does not have a d2-retraction sequence, append* $d_c :=$ *NO to* $\mathscr{D}$*.*
     - 4.3.2 *If* Q_c *has a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$*, append* $d_c :=$ *TRUE to* $\mathscr{D}$ *and proceed as follows. For every* $i \in \{1, \ldots, |Q_c| - 1\}$ *so that, in* $Q_c \setminus \{a_1, \ldots, a_{i-1}\}$*,* a_i *is a minimal element that is not irreducible, but retractable to the minimal element* a_{i+j} *of* $Q_c \setminus \{a_1, \ldots, a_{i-1}\}$*, check if* $\uparrow a_i \cap \uparrow a_{i+j}$ *is already listed in* $\mathscr{S}$*. If not, append* $\uparrow a_i \cap \uparrow a_{i+j}$ *to* $\mathscr{S}$*.*

3. *If* $\mathscr{S} = (Q_1, \ldots, Q_m)$ *and* $c > m$*, return* $\mathscr{S}$, $\mathscr{D}$*, exit.*
4. *Given* $\mathscr{S} = (Q_1, \ldots, Q_m)$*, consider the set* Q_c*.*
   - 4.1 *If* Q_c *is a singleton, append* $d_c :=$ *TRUE to* $\mathscr{D}$*.*
   - 4.2 *If* Q_c *is an antichain and* $|Q_c| > 1$*, append* $d_c :=$ *AC to* $\mathscr{D}$*.*
   - 4.3 *If* Q_c *is not an antichain and* $|Q_c| > 1$*, execute an algorithm to find a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$ *for* Q_c*.*
     - 4.3.1 *If* Q_c *does not have a d2-retraction sequence, append* $d_c :=$ *NO to* $\mathscr{D}$*.*
     - 4.3.2 *If* Q_c *has a d2-retraction sequence* $a_1, \ldots, a_{|Q_c|}$*, append* $d_c :=$ *TRUE to* $\mathscr{D}$ *and proceed as follows. For every* $i \in \{1, \ldots, |Q_c| - 1\}$ *so that, in* $Q_c \setminus \{a_1, \ldots, a_{i-1}\}$*,* a_i *is a minimal element that is not irreducible, but retractable to the minimal element* a_{i+j} *of* $Q_c \setminus \{a_1, \ldots, a_{i-1}\}$*, check if* $\uparrow a_i \cap \uparrow a_{i+j}$ *is already listed in* $\mathscr{S}$*. If not, append* $\uparrow a_i \cap \uparrow a_{i+j}$ *to* $\mathscr{S}$*.*
5. *Increase* c *by* 1 *and continue at 3.*

Theorem.

Theorem. *Let P be a finite ordered set.*

Theorem. *Let P be a finite ordered set. Then* **D2CCcheck** *applied to P terminates in polynomial time.*

Theorem. *Let P be a finite ordered set. Then* **D2CCcheck** *applied to P terminates in polynomial time. Moreover,*

Theorem. *Let P be a finite ordered set. Then* **D2CCcheck** *applied to P terminates in polynomial time. Moreover,*

1. *P is d2-connectedly collapsible iff all entries of $\mathcal{D}$ are TRUE.*

Theorem. *Let P be a finite ordered set. Then* **D2CCcheck** *applied to P terminates in polynomial time. Moreover,*

1. *P is d2-connectedly collapsible iff all entries of $\mathscr{D}$ are TRUE.*
2. *P is d2-collapsible, but not d2-connectedly collapsible iff*

Theorem. *Let P be a finite ordered set. Then* **D2CCcheck** *applied to P terminates in polynomial time. Moreover,*

1. *P is d2-connectedly collapsible iff all entries of $\mathscr{D}$ are TRUE.*
2. *P is d2-collapsible, but not d2-connectedly collapsible iff all entries of $\mathscr{D}$ are TRUE or AC.*

Theorem. *Let P be a finite ordered set. Then* **D2CCcheck** *applied to P terminates in polynomial time. Moreover,*

1. *P is d2-connectedly collapsible iff all entries of $\mathscr{D}$ are TRUE.*
2. *P is d2-collapsible, but not d2-connectedly collapsible iff all entries of $\mathscr{D}$ are TRUE or AC.*
3. *P is not d2-collapsible iff $\mathscr{D}(P)$ has an entry NO.*

Final Comments

Final Comments

1. This idea generalizes, but not to all collapsible ordered sets.

Final Comments

1. This idea generalizes, but not to all collapsible ordered sets. The problem is that the sets that need to be checked recursively can, in general, be of the form $(\updownarrow a_1 \cap \updownarrow b_1)$

Final Comments

1. This idea generalizes, but not to all collapsible ordered sets. The problem is that the sets that need to be checked recursively can, in general, be of the form $(\updownarrow a_1 \cap \updownarrow b_1) \cap (\updownarrow a_2 \cap \updownarrow b_2)$

Final Comments

1. This idea generalizes, but not to all collapsible ordered sets. The problem is that the sets that need to be checked recursively can, in general, be of the form $(\updownarrow a_1 \cap \updownarrow b_1) \cap (\updownarrow a_2 \cap \updownarrow b_2) \cap \cdots \cap (\updownarrow a_m \cap \updownarrow b_m)$.

Final Comments

1. This idea generalizes, but not to all collapsible ordered sets. The problem is that the sets that need to be checked recursively can, in general, be of the form $(\updownarrow a_1 \cap \updownarrow b_1) \cap (\updownarrow a_2 \cap \updownarrow b_2) \cap \cdots \cap (\updownarrow a_m \cap \updownarrow b_m)$.
2. Let P be a finite ordered set that does not contain any crowns with more than 4 elements.

Final Comments

1. This idea generalizes, but not to all collapsible ordered sets. The problem is that the sets that need to be checked recursively can, in general, be of the form $(\updownarrow a_1 \cap \updownarrow b_1) \cap (\updownarrow a_2 \cap \updownarrow b_2) \cap \cdots \cap (\updownarrow a_m \cap \updownarrow b_m)$.
2. Let P be a finite ordered set that does not contain any crowns with more than 4 elements. Must P be d2-collapsible?