# Coefficients of the Laurent expansion of the Hilbert series of Gorenstein rings

# Christopher Seaton, Rhodes College joint work with Hans-Christian Herbig and Daniel Herden

4th annual Mississippi Discrete Mathematics Workshop

University of Mississippi, Oxford, MS

November 15th, 2015

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Let  $R = \bigoplus_{k=0}^{\infty} R_k$  be a finitely generated graded algebra over a field  $\mathbb{K} = R_0$ . Let

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The Hilbert series converges on |t| < 1 and can be expressed in the form

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If dim R > 0, then Hilb<sub>R</sub>(t) has a pole at t = 1 and hence admits a Laurent expansion

$$\mathsf{Hilb}_{R}(t) = \sum_{k=0}^{\infty} \gamma_{k} (1-t)^{k-\dim R} = \frac{\gamma_{0}}{(1-t)^{\dim R}} + \frac{\gamma_{1}}{(1-t)^{\dim R-1}} + \cdots$$

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$$\frac{2\gamma_1}{\gamma_0} = \#\{\text{pseudoreflections in } G\},\$$

where a **pseudoreflection** is a  $g \in G$  such that  $V^g$  has codimension 1.

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A rational function  $\psi(t)$  is symplectic at  $a \in \mathbb{C}$  of order  $d \in \mathbb{Z}$  if

$$x^d\psi(a-x)\in\mathbb{C}[\![x]\!]$$

is symplectic.

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**Theorem** (Herbig–Herden–S., 2015) For a formal power series  $\varphi(x) \in \mathbb{C}[\![y]\!] = \sum_{k=0}^{\infty} \gamma_k x^k$ , the following are equivalent.

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The **shift** of a Gorenstein algebra R is r := -(d + a(R)), hence d = -a(R) corresponds to a shift of 0.

**Corollary** A graded Cohen-Macaulay algebra R is Gorenstein with shift 0 if and only if  $Hilb_R(t)$  is symplectic of order dim R at t = 1.

C. Seaton (Rhodes College)

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**Definition** The Euler polynomials  $E_n(x)$ ,  $n \ge 0$  are defined by

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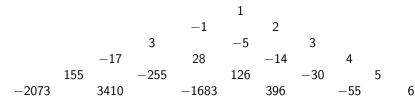
$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

Define  $\begin{bmatrix} n \\ k \end{bmatrix}$  by

$$x(x^{2n}-E_{2n}(x)) =: \sum_{k=0}^{2n} {n \brack i} x^{2k},$$

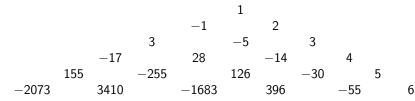
i.e.  $\begin{bmatrix}n\\k\end{bmatrix}$  is the kth odd degree coefficient of the (2n)th Euler polynomial.

The first six lines of the triangle of the  $\begin{bmatrix} n \\ k \end{bmatrix}$ :



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**Proposition** Define

$$\begin{split} \psi_n(x) &:= \frac{1}{(2n-1)!} \sum_{k=0}^{\infty} (-1)^{k-1} E_{k-1}^{(2n-1)}(0) \, x^k \\ &= -x^{2n} - \sum_{k=n}^{\infty} {k \brack n} x^{2k+1}. \end{split}$$

Then the  $\psi_n(x)$ ,  $n \ge 0$  form a symplectic basis.

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# Even Coefficients Determine the Odd (Shift 0)

**Theorem** (Herbig–Herden–S., 2015) Let  $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^k$  be a formal power series. Then  $\varphi(x)$  is symplectic if and only if for each  $n \ge 0$ ,

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**Corollary** For all integers  $n, k, \ell$  we have

$$\begin{bmatrix} n-k\\ \ell \end{bmatrix} + \begin{bmatrix} n-\ell\\ k \end{bmatrix} = \begin{bmatrix} n\\ k+\ell \end{bmatrix} + \sum_{i} \sum_{r} \begin{bmatrix} n\\ i \end{bmatrix} \begin{bmatrix} r-1\\ k \end{bmatrix} \begin{bmatrix} i-r\\ \ell \end{bmatrix}.$$

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For the power series  $\varphi(x) := x^d \operatorname{Hilb}_R(1-x)$ , this corresponds to

$$\varphi\left(\frac{x}{x-1}\right) = (1-x)^r \varphi(x).$$

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#### Shift r > 0

**Theorem** (Herbig–Herden–S.) A power series  $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^d$  satisfies

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for some r > 0 if and only if, for each  $m \ge 1$ ,

$$\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}\gamma_{m+i-r}=0, \hspace{0.1in} ext{if } r \hspace{0.1in} ext{is even,}$$

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 if r is even,

$$\sum_{i=0}^{m} (-1)^{i} \binom{2m+r-2}{m-i} \binom{m+i}{i} \gamma_{m+i-1} = 0, \text{ if } r \text{ is odd.}$$

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### Shift r = 1

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#### Shift r = 1

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Dividing by common factors in each row, the resulting nonzero coefficients are the **Lucas triangle:** 

# Shift r > 0 odd

*r* = 3 :

3	2	0								
0	10	15	6	0						
0	0	35	84	70	20	0				
0	0	0	126	420	540	315	70	0		
0	0	0	0	462	1980	3465	3080	1386	252	0

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#### Shift r > 0 odd

r = 3: 0 . . . r = 5:

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for some r < 0 if and only if, for  $1 \le m \le \lceil -r/2 \rceil$ ,

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$$\sum_{i=0}^{m-1} (-1)^i \binom{2m-1}{m-i} \binom{m+i}{i} \gamma_{m+i-r+1} = 0, \quad \text{if } r \text{ is odd.}$$

## Shift r = -6

Or:

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## Shift r = -5

Or:

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Then  $\mathcal{F}_0$  is algebra-generated by  $x^2/(1-x)$ , and each  $\mathcal{F}_r$  is a  $\mathcal{F}_0$ -module generated by  $(x-2)^r$ .

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Hence  $\mathcal{F} = \bigoplus_{r \in \mathbb{Z}} \mathcal{F}_r$  is a  $\mathbb{Z}$ -graded algebra generated by  $x^2/(1-x)$  and x-2.

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## Shift $r \neq 0$ : Even Coefficients Determine the Odd

**Definition** Define  $\binom{n}{k}$  by

$$x(x^{2n+1}-E_{2n+1}(x)) =: \sum_{k=0}^{2n+1} {n \choose k} x^{2k},$$

*i.e.*  $\binom{n}{k}$  is the kth even degree coefficient of the (2n+1)st Euler polynomial.

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$$\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^k \in \mathcal{F}_r$$
 if and only if:

$$\gamma_{2n+1} = \sum_{k=0}^{n} {n+m \choose k+m} \gamma_{2k} \qquad n \ge 0, \qquad r = 2m+1 > 0$$

**Theorem (Herbig–Herden–S.)**  $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^k \in \mathcal{F}_r$  if and only if:  $\gamma_{2n+1} = \sum_{k=0}^n {n+m \\ k+m} \gamma_{2k}$   $n \ge 0,$  r = 2m + 1 > 0, $\gamma_{2n-2m+1} = \sum_{k=0}^n {n \\ k} \gamma_{2k-2m}$   $n \ge 0,$  r = 2m > 0

 $\varphi(x) = \sum_{k=1}^{\infty} \gamma_k x^k \in \mathcal{F}_r$  if and only if: **Theorem** (Herbig–Herden–S.)  $\gamma_{2n+1} = \sum_{k=0}^{n} \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\} \gamma_{2k}$ r=2m+1>0,n > 0.  $\gamma_{2n-2m+1} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \gamma_{2k-2m}$ n > 0, r = 2m > 0,  $\gamma_{2n+2k+1} = \sum_{k=0}^{n} {n \\ k} \gamma_{2m+2k},$ n > 0. and  $\gamma_{2n+1} = -\sum_{i=1}^{n} \left\{ \frac{m-k-1}{m-n-1} \right\} \gamma_{2k}, \quad 0 \le n \le m-1, \quad r = -2m+1 < 0$ 

 $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^k \in \mathcal{F}_r$  if and only if: **Theorem** (Herbig–Herden–S.)  $\gamma_{2n+1} = \sum_{l=0}^{n} \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\} \gamma_{2k}$ n > 0. r = 2m + 1 > 0.  $\gamma_{2n-2m+1} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \gamma_{2k-2m}$ n > 0, r = 2m > 0,  $\gamma_{2n+2k+1} = \sum_{k=0}^{n} {n \\ k} \gamma_{2m+2k},$ n > 0. and  $\gamma_{2n+1} = -\sum_{k=0}^{n} {m-k-1 \choose m-n-1} \gamma_{2k},$  $0 \le n \le m - 1$ , r = -2m + 1 < 0,  $\gamma_{2m+2n+1} = \sum_{k=0}^{n} {n \choose k} \gamma_{2m+2k},$ n > 0. and  $\gamma_{2n+1} = -\sum_{k=0}^{n} {m-k \brack {m-n}} \gamma_{2k},$  $0 \le n \le m - 1$ , r = -2m < 0.

# Thank you!

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