Multiplicative Zagreb indices of Cactus graphs

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Outline

- Introduction

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  - Cactus graphs
  - Zagreb Index and Multiplicative Zagreb Index

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- Some proofs

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Why cactus?

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- Lin et al.(2007), Liu and Lu (2008) obtained some sharp bounds of several chemical indices of cactus graphs, such as Wiener index, Merrifield-Simmons index, Hosoya index and Randić index.
- In 2012, Li et al. gave the upper bounds on Zagreb indices of cactus graphs and lower bounds of cactus graph with at least one cycle.

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- Wang and Tan(2015) characterized the extremal cacti having the largest Wiener and hyper-Wiener indices among the class $\mathcal{C}_{n,k}$.
- Wang and Kang(2015) found the extremal bounds of another chemical index of Harary index for the cacti $\mathcal{C}_{n,k}$.

What something else could be found on cactus graphs?

Zagreb Indices

Definition (Gutman and Trinajsti 1972)

The first and second **Zagreb indices** of the graph $G = (V, E)$ are defined as

$$M_1 = \sum_{v \in V(G)} d(v)^2; M_2 = \sum_{uv \in E(G)} d(u)d(v).$$

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The first **generalized** and second **Multiplicative Zagreb indices** of graph $G = (V, E)$ are defined as follows: for any real number $c > 0$,

$$\prod_{1,c}(G) = \prod_{v \in V(G)} d(v)^c;$$

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$$\prod_2(G) = \prod_{uv \in E(G)} d(u)d(v) = \prod_{v \in V(G)} d(v)^{d(v)}.$$

- For general graphs, a lower bound of a chemical index, Randić index $= \sum_{uv \in E(G)} [d(u)d(v)]^{-\frac{1}{2}}$, was given by Bollob$\acute{a}$s and Erd$\ddot{o}$s(1998), while an upper bound was presented by Lu. et al in 2004.

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- In 2004, Das applied the minimal and maximal degree to obtain the upper bound for the sum of the squares of the degrees of a graph, the first Zagreb index.
- In 2010, Zhao and Li provided the maximal Zagreb index of graphs with k cut vertices.

History of Zagreb indices

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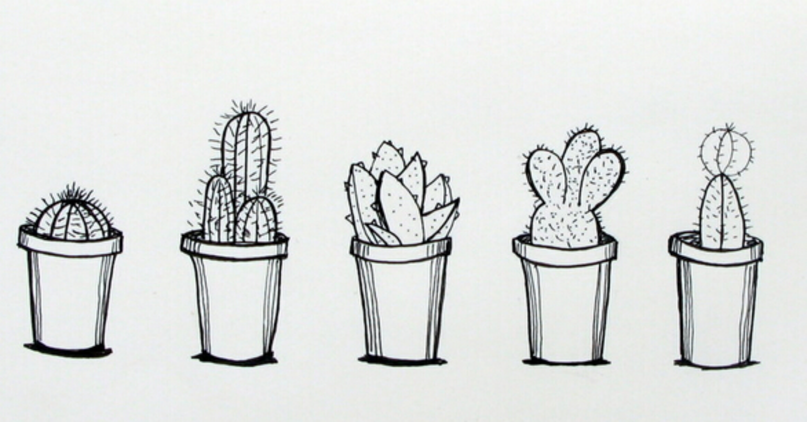
- Wang and Wei (2015) gave the sharp upper and lower bounds of Generalized Multiplicative Zagreb indices of k-trees, respectively.
- $\cdots\cdots$

Our results

- Li et al.(2012) gave some bounds on Zagreb indices of cactus graphs.
- Now we obtained the sharp upper and lower bounds of Multiplicative Zagreb indices for cactus graphs and characterize the graphs achieved such bounds.



Theorem (Wang and Wei 2016+)

For any graph G in $\mathcal{C}_n^k$,

$$\prod_{1,c}(G) \geq \begin{cases} 3^{kc}2^{(n-2k)c} & \textit{if} \quad k = 0, 1, \\ 2^{(n-k-1)c}k^c & \textit{if} \quad k \geq 2. \end{cases}$$

the equalities hold if and only if their degree sequences are $\underbrace{3, 3, ..., 3}_{k}, \underbrace{2, 2, ..., 2}_{n-2k}, \underbrace{1, 1, ..., 1}_{k}$ *and* $k, \underbrace{2, 2, ..., 2}_{n-k-1}, \underbrace{1, 1, ..., 1}_{k}$, *respectively.*

Theorem 2

Theorem (Wang and Wei 2016+)

For any graph G in $\mathcal{C}_n^k$ with $n \leq k+3$,

$$\prod_{1,c}(G) \leq \begin{cases} k^c & if \quad n = k+1, \\ (\lceil \frac{k}{2} \rceil + 1)^c (\lfloor \frac{k}{2} \rfloor + 1)^c & if \quad n = k+2, \\ (\lceil \frac{k}{3} \rceil + 2)^c (\lfloor \frac{k}{3} \rfloor + 2)^c (k - \lceil \frac{k}{3} \rceil - \lfloor \frac{k}{3} \rfloor + 2)^c & if \quad n = k+3, \end{cases}$$

the equalities hold if and only if their degree sequences are $k, \underbrace{1, 1, ..., 1}_{k}$; $\lceil \frac{k}{2} \rceil + 1, \lfloor \frac{k}{2} \rfloor + 1, \underbrace{1, 1, ..., 1}_{k}$ *and* $\lceil \frac{k}{3} \rceil + 2, \lfloor \frac{k}{3} \rfloor + 2, k - \lceil \frac{k}{3} \rceil - \lfloor \frac{k}{3} \rfloor + 2, \underbrace{1, 1, ..., 1}_{k}$, *respectively.*

Theorem 3

Theorem (Wang and Wei 2016+)

For any graph G in $\mathcal{C}_n^k$ with $n \geq k+4$ and $t \geq 0$,

$$\prod_{1,c}(G) \leq \begin{cases} 16^c & \text{if} \quad k=0, n=4, \\ 2^{(3t+6)c} & \text{if} \quad k=0, n=2t+5, \\ 2^{(3t+4)c}9^c & \text{if} \quad k=0, n=2(t+3), \end{cases}$$

the equalities hold if and only if their degree sequences are $2,2,2,2$; $\underbrace{4,4,...,4}_{t+1},\underbrace{2,2,...,2}_{t+4}$ *and* $\underbrace{4,4,...,4}_{t},3,3,\underbrace{2,2,...,2}_{t+4}$, *respectively;*

For $k \neq 0$, if $\prod_{1,c}(G)$ attains the maximal value, then one of the following statements holds: For any nonpendant vertices u, v, either (i) $|d(u)-d(v)| \leq 1$ or (ii) $d(u) \in \{2,3,4\}$ and G contains no cycles of length greater than 3, no dense paths of length greater than 1 except for at most one of them with length 2, no paths of length greater 0 that connects only two cycles except for at most one of them with length 1.

Theorem 4

Theorem (Wang and Wei 2016+)

For any graph G in $\mathcal{C}_n^k$ with $\gamma = \frac{k-2}{n-k}$,

$$\prod_2(G) \geq \begin{cases} 3^{3k}2^{2(n-2k)} & \text{if} \quad k = 0, 1, \\ (2+\lceil\gamma\rceil)^{(2+\lceil\gamma\rceil)[k-2-\lfloor\gamma\rfloor(n-k)]} & \\ (2+\lfloor\gamma\rfloor)^{(2+\lfloor\gamma\rfloor)[n-2k+2+\lfloor\gamma\rfloor(n-k)]} & \text{if} \quad k \geq 2, \end{cases}$$

the equalities hold if and only if their degree sequences are $\underbrace{3,3,...,3}_{k},\underbrace{2,2,...,2}_{n-2k},\underbrace{1,1,...,1}_{k}$ *and* $\underbrace{2+\lceil\gamma\rceil,2+\lceil\gamma\rceil,...,2+\lceil\gamma\rceil}_{k-2-\lfloor\gamma\rfloor(n-k)},\underbrace{2+\lfloor\gamma\rfloor,2+\lfloor\gamma\rfloor,...,2+\lfloor\gamma\rfloor}_{n-2k+2+\lfloor\gamma\rfloor(n-k)},\underbrace{1,1,...,1}_{k},$ *respectively.*

Theorem 5

Theorem (Wang and Wei 2016+)

For any graph G in $\mathcal{C}_n^k$,

$$\prod_2(G) \leq \begin{cases} (n-2)^{n-2}2^{2(n-k-1)} & \text{if} \quad n-k \equiv 0 (mod\, 2), \\ (n-1)^{n-1}2^{2(n-k-1)} & \text{if} \quad n-k \equiv 1 (mod\, 2), \end{cases}$$

the equalities hold if and only if their degree sequences are $n-2, \underbrace{2, 2, ..., 2}_{n-k-1}, \underbrace{1, 1, ..., 1}_{k}$ *and* $n-1, \underbrace{2, 2, ..., 2}_{n-k-1}, \underbrace{1, 1, ..., 1}_{k},$ *respectively.*

Analytic Tool

- The function $f(x) = \dfrac{x}{x+m}$ is strictly increasing for $x \in [0, \infty)$, where m is a positive integer.
- The function $g(x) = \dfrac{x^x}{(x+m)^{x+m}}$ is strictly decreasing for $x \in [0, \infty)$, where m is a positive integer.

Sketch Proof of Theorem 1

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- Choose a graph G in $\mathcal{C}_n^k$ such that $\prod_{1,c}(G)$ achieves the minimal value.
- If $k = 0$ or 1, then G must be an unicyclic graph.

Why G would be unicyclic?

- If G is not unicyclic, after the transforming, obtain a new graph G' such that $\prod_{1,c}(G') < \prod_{1,c}(G)$, a contradiction to the choice of G.

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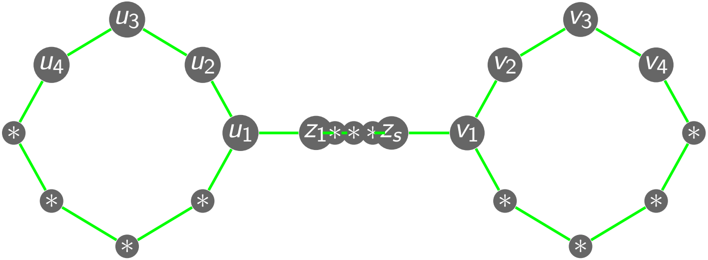

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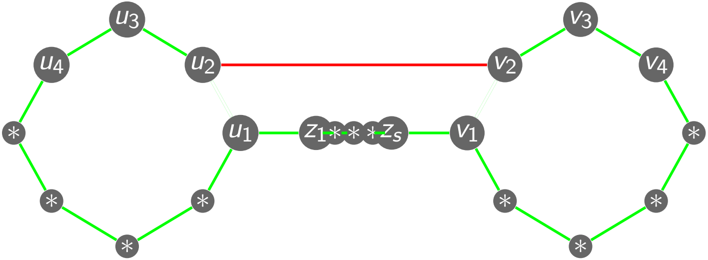

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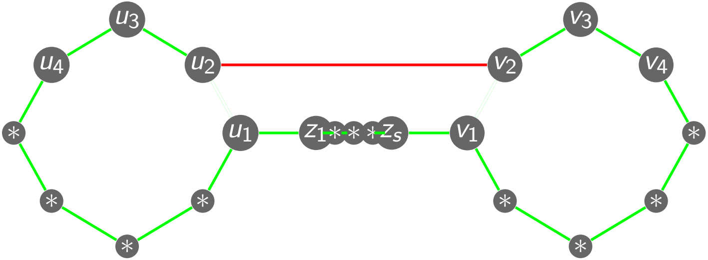

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- If $k \geq 2$, by the choice of G, we see that G is a tree.

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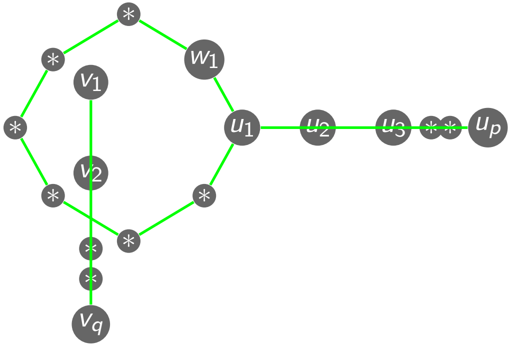

- For $k = 2$, G is a path, that is, the degree sequence of G is $\underbrace{2, 2, \ldots, 2}_{n-2}, 1, 1$, done. Next we consider $k \geq 3$.

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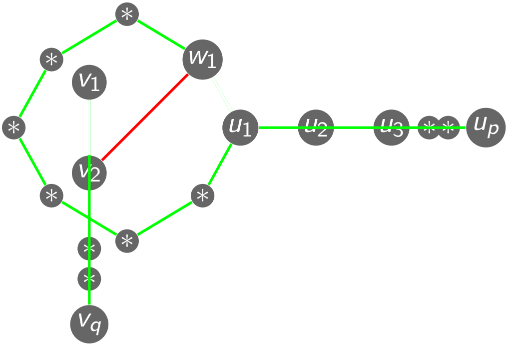

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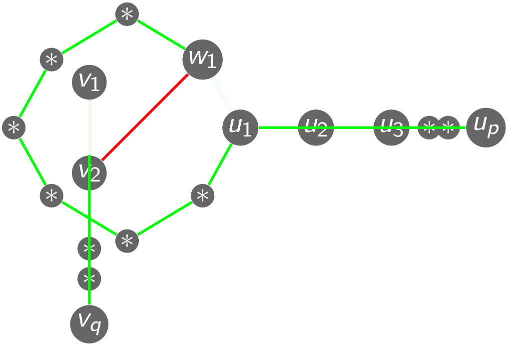

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- $d(v) \leq k$ for any $v \in V(G)$. Otherwise, if there is a vertex v with $d(v) \geq k+1$, since G is a tree, then G has more than k pendant vertices, a contradiction to the choice of G.

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- Let v be the vertex with maximal degree Δ, if $\Delta = k$, then $G - v$ is a set of paths. Otherwise, there exists a vertex $u \in V(G) - \{v\}$ such that $d(u) \geq 3$ and since G is a tree, then G contains more than k pendant vertices, a contradiction to the choice of G. Thus, the degree sequence of G is $k, \underbrace{2, 2, ..., 2}_{n-k-1}, \underbrace{1, 1, ..., 1}_{k}$.

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- If $\Delta < k$, then G contains at least 2 cut vertices, say $u_1, u_2, ..., u_t$, such that $G - u_i$ has at least 3 components with $i \in [1, t]$ and $t \geq 2$. Otherwise, since G is a tree, G only contains $\Delta < k$ pendant vertices, a contradiction.

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- Let $P = w_1 w_2 ... w_{s'} w_s$ be a path of $G - \{u_1, u_2, ..., u_t\}$ such that $w_s \in \{u_1, u_2, ..., u_t\} - \{v\}$ and P contains only a unique pendant vertex w_1.

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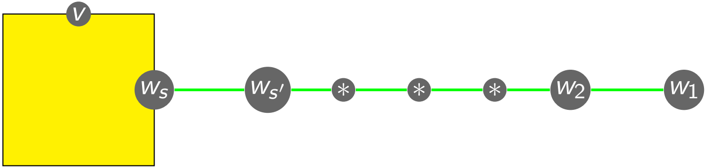

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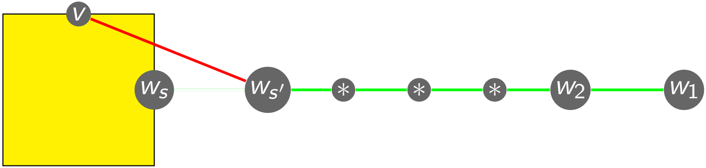

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Using analytic tool

- $d_{G'}(v) = d(v) + 1 = \Delta + 1$ and $d_{G'}(w_s) = d(w_s) - 1$, since $G' = (G - \{w_{s'} w_s\}) \cup \{w_{s-1} v\}$.

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- By the first analytic tool, we have

$$\frac{\prod_{1,c}(G)}{\prod_{1,c}(G')} = \frac{\Delta^c d(w_s)^c}{(\Delta + 1)^c (d(w_s) - 1)^c} = \frac{\frac{\Delta^c}{(\Delta+1)^c}}{\frac{[d(w_s)-1]^c}{d(w_s)^c}} > 1,$$

a contradiction with the choice of G.

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- Once the maximal degree of G' is still less than k, then we can continue this process until $\Delta = k$, thus we can find the desired graph with the degree sequence of $k, \underbrace{2, 2, ..., 2}_{n-k-1}, \underbrace{1, 1, ..., 1}_{k}$.

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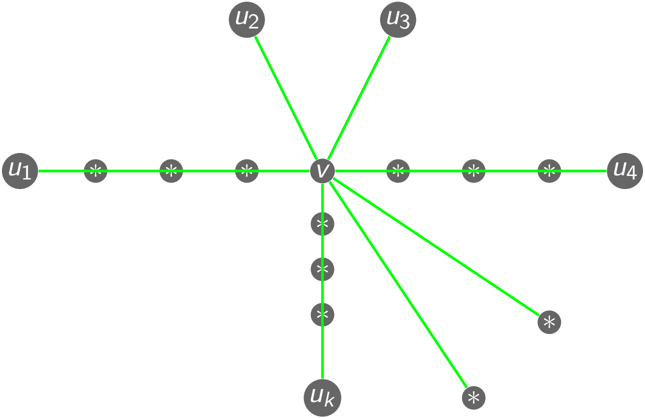

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- If $\prod_{1,c}(G)$ attains the maximal, by *"Arithmetic-Mean and Geometric-Mean inequality: $x_1x_2...x_n \leq (\frac{x_1+x_2+...+x_n}{n})^n$, the equality holds if and only if $x_1 = x_2 = ... = x_n$"*, it is not hard to obtain our result.

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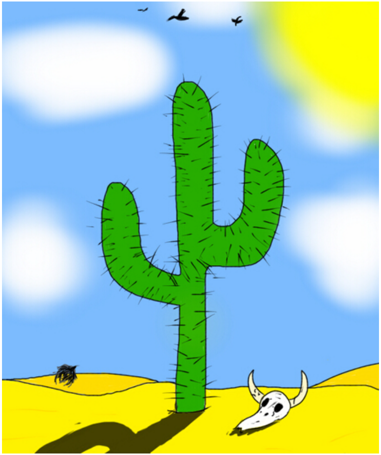
- By some transforming and using the analytic tools, we can always find a new graph such that contradicted with the assumptions.
- Done.



Thank you