Unavoidable Immersions of Large 3-Edge-Connected Graphs

Matt Barnes

Louisiana State University

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Previous Work and Motivations

Theorem (Ramsey)

There is a function f such that, for every integer $k \geq 1$, every graph of order at least $f(k)$ contains a induced, complete subgraph on k vertices, or an independent set of vertices of size k.

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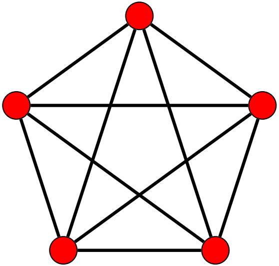

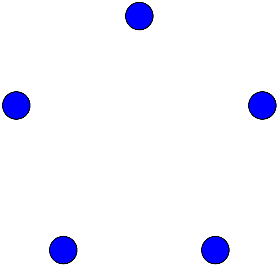

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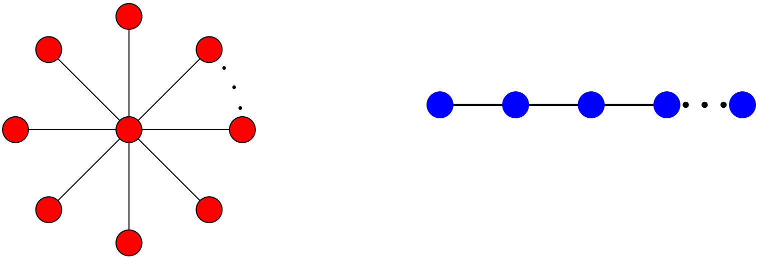

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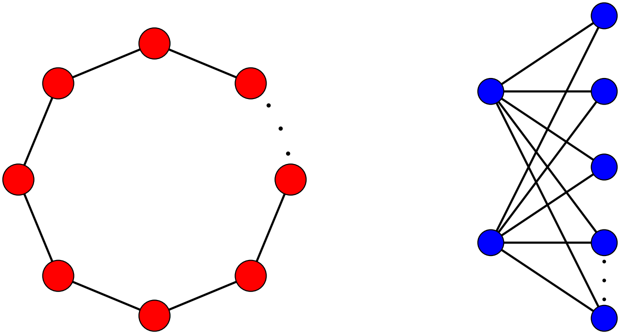

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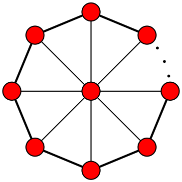

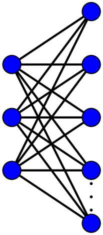

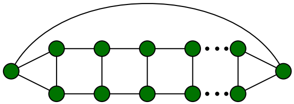

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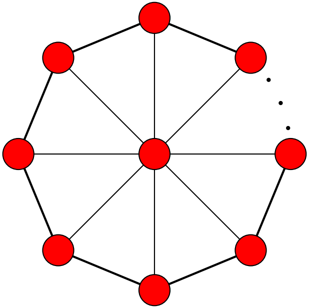

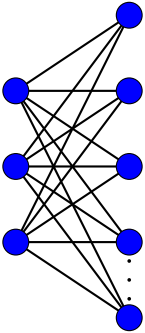

Topological Minor to Immersion

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A graph H is a topological minor of a graph G if there is a one to one map $\tau : H \to G$ such that adjacent vertices of H are mapped to vertices of G connected by mutually internally vertex disjoint paths.

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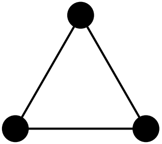

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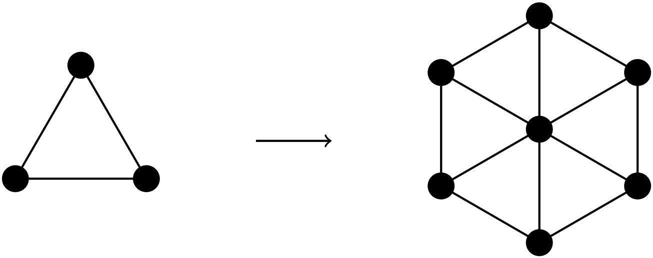

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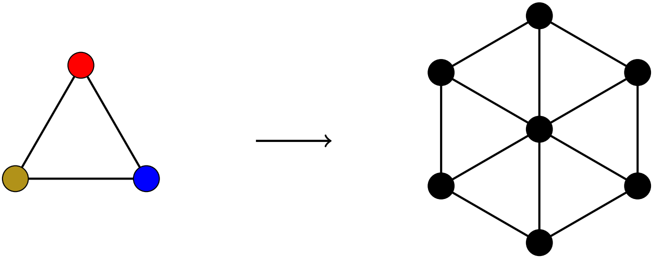

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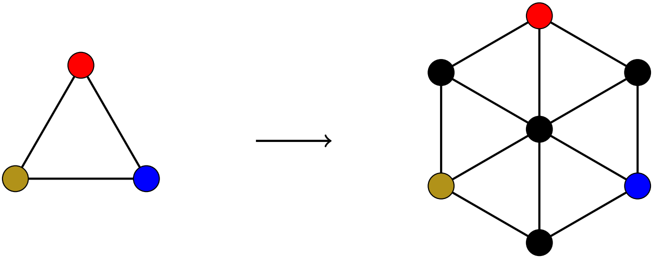

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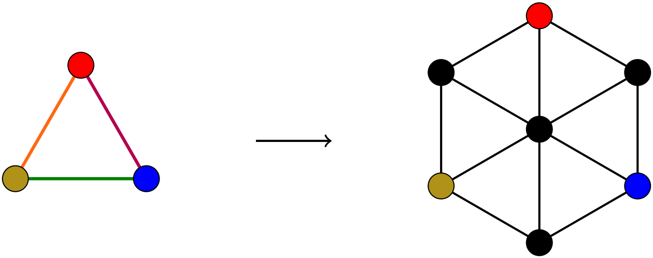

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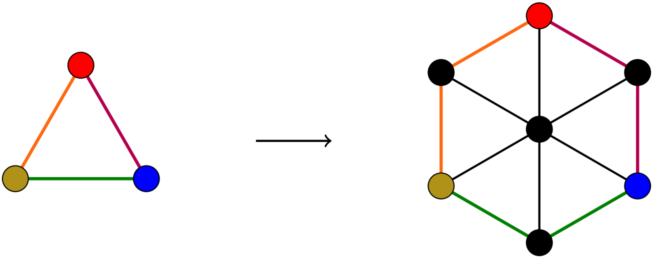

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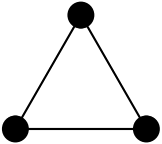

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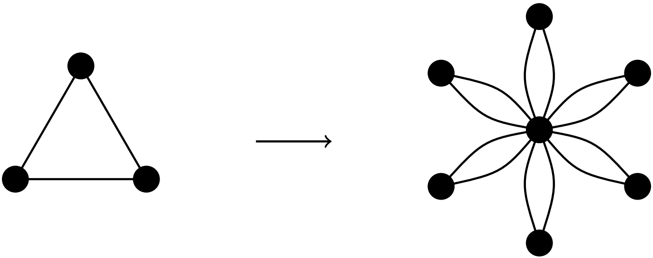

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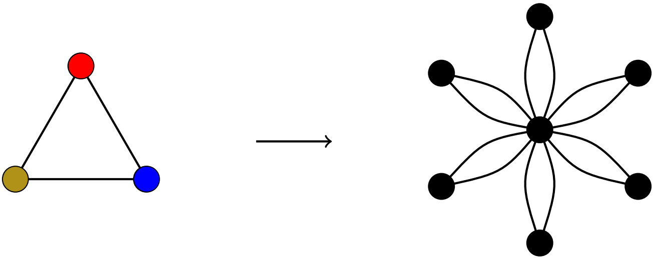

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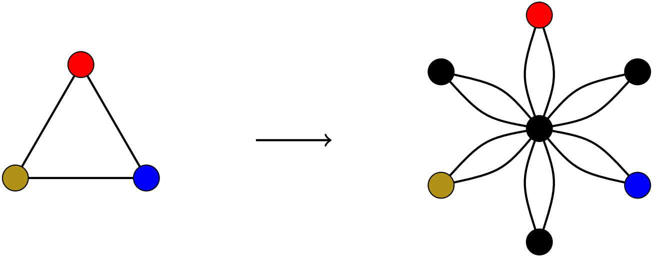

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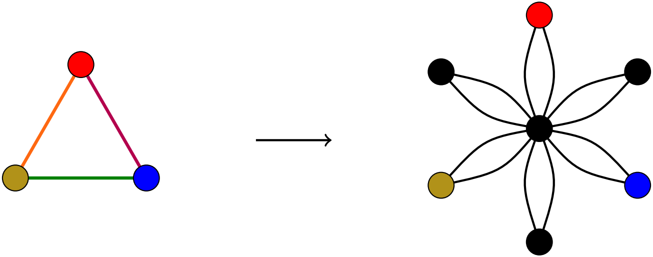

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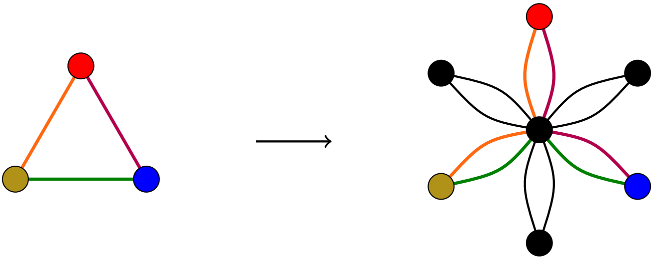

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Let P be a path in a graph G. A **lifting** of P consists of deleting the edge set of P, and adding an edge between the distinct end-vertices of P.

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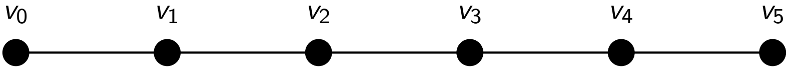

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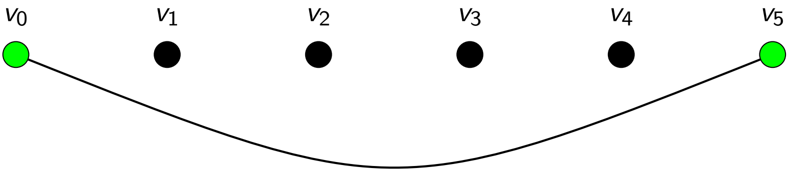

Immersions

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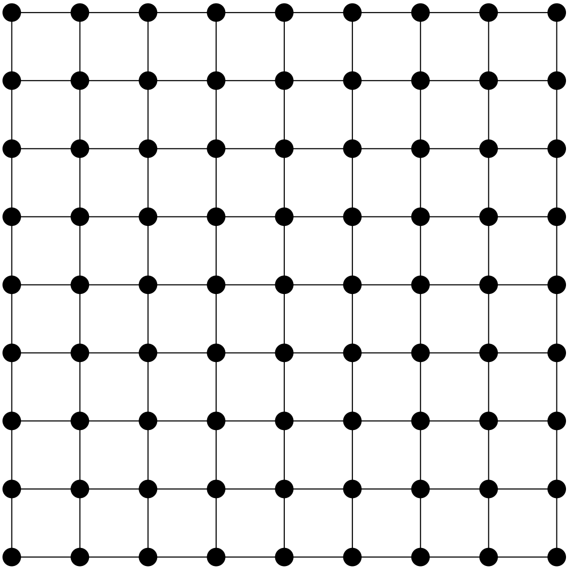
A graph H is immersed in a graph G if H can be obtained from G by a series of:

- vertex deletions
- edge deletions
- liftings

Immersion Example



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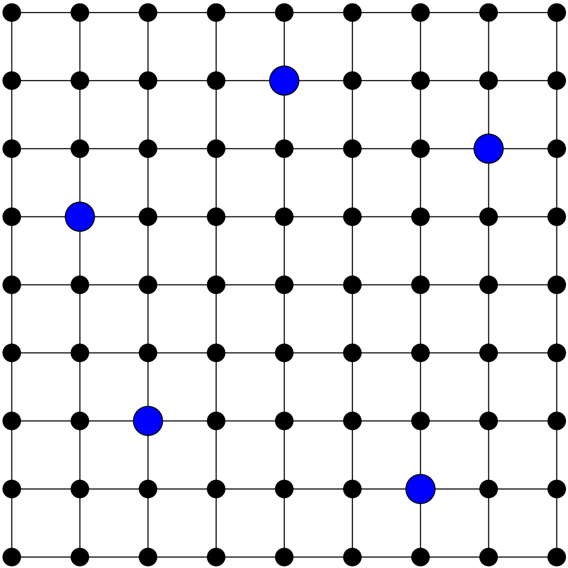

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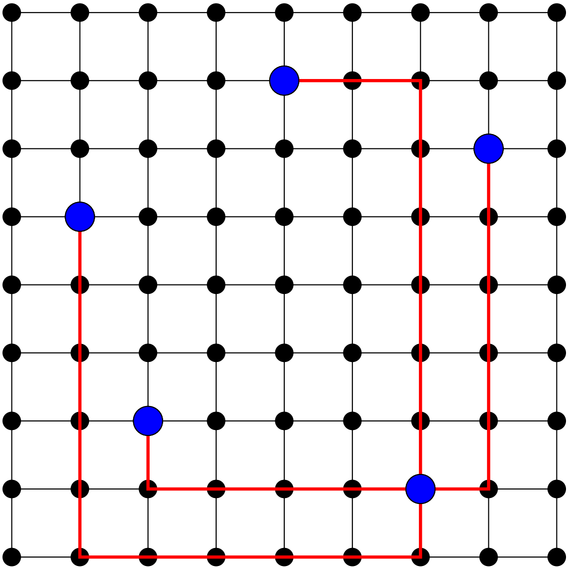

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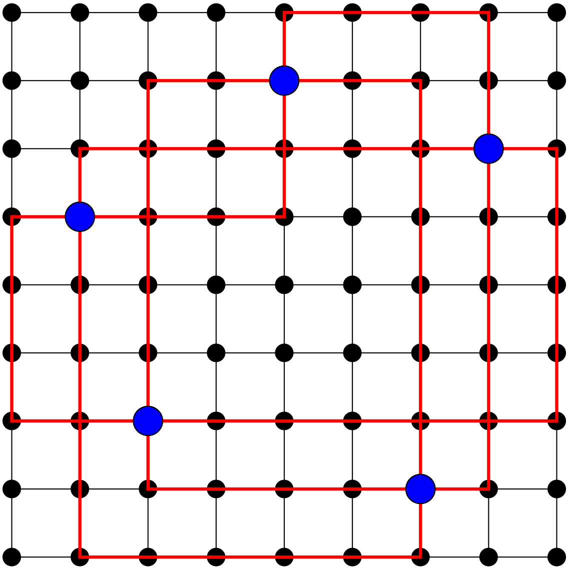

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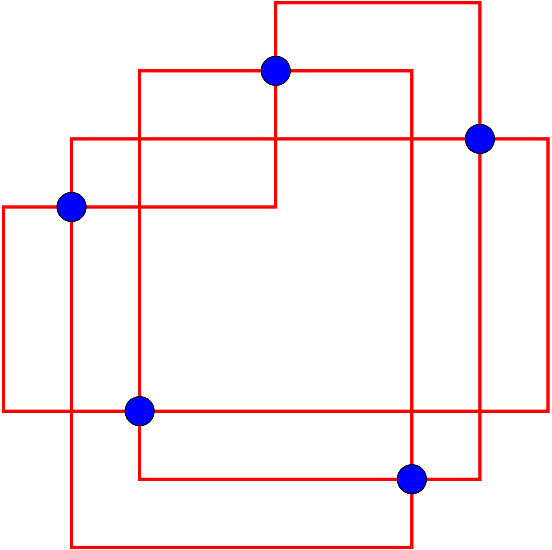

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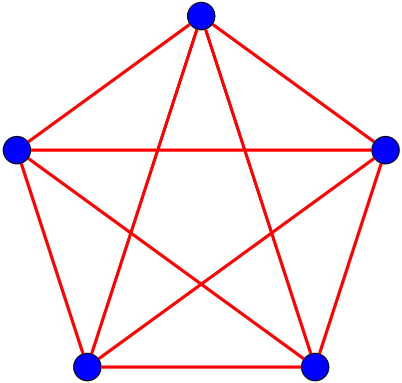

Main Theorem

Theorem (B., Oporowski)

There is a function f such that, for every integer $k \geq 3$, every 3-edge-connected graph with order at least $f(k)$ admits an immersion of $S_{3,k}$, L_k^+, or $P_{2,k}^+$.

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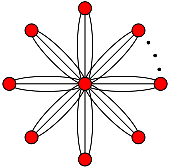
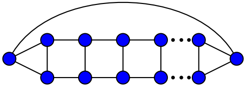
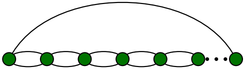


$S_{3,k}$



L_k^+



$P_{2,k}^+$

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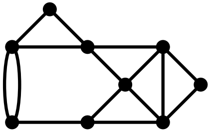

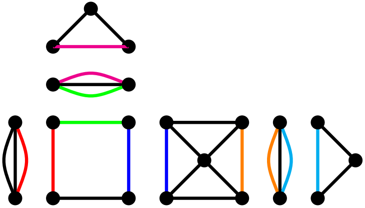

Proof sketch

G is a “large” 3-edge-connected graph

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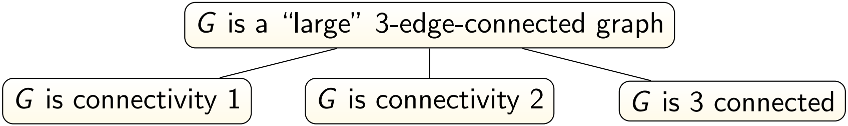

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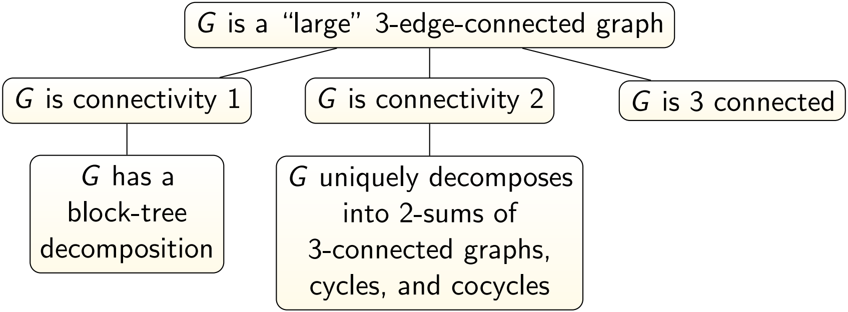

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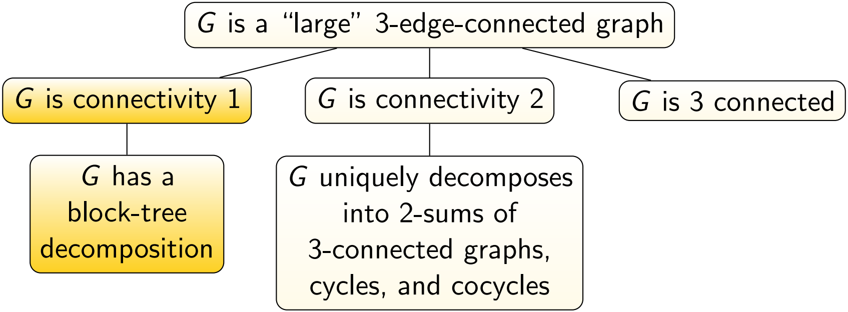

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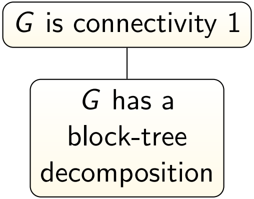

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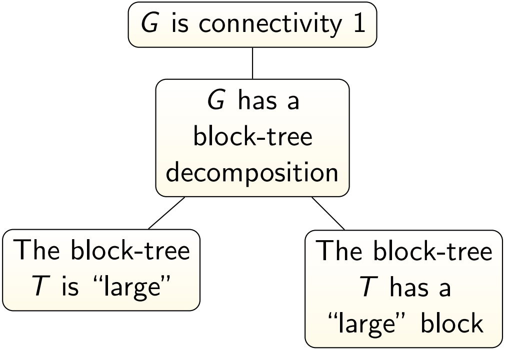

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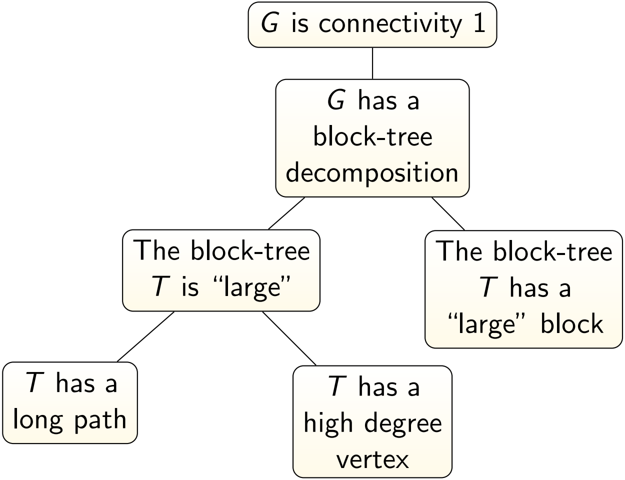

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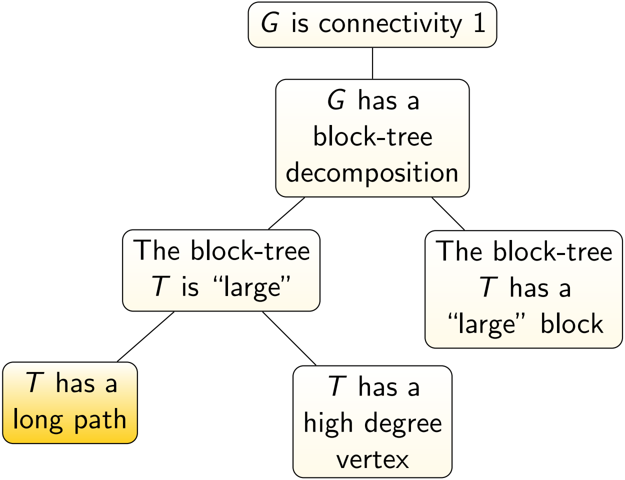

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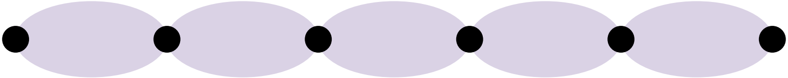

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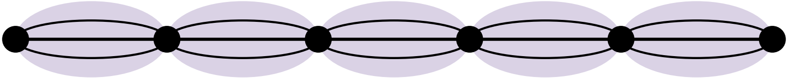

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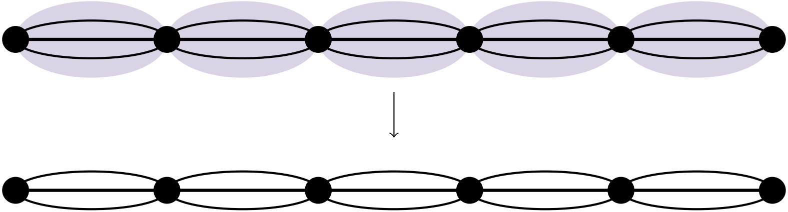

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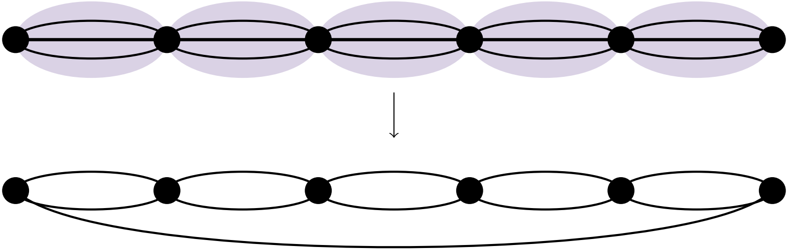

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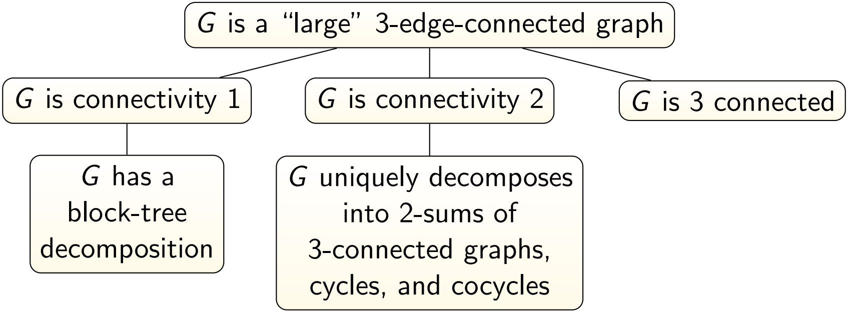

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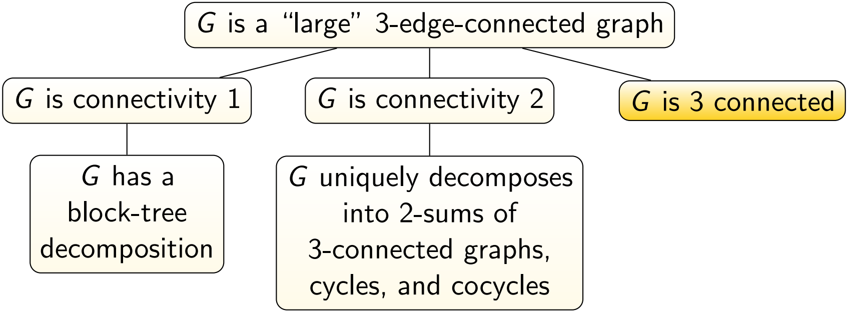

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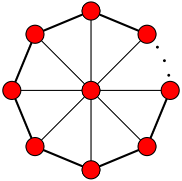
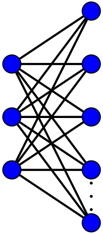
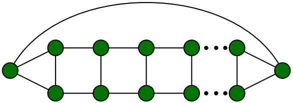

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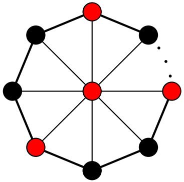

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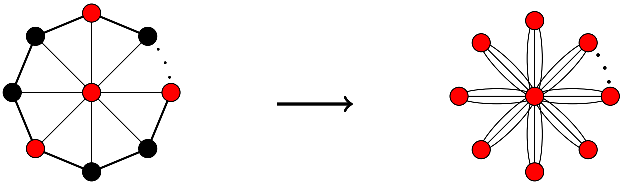

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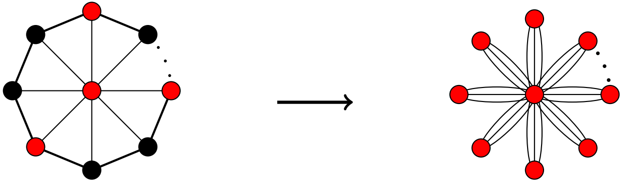

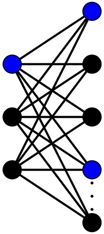

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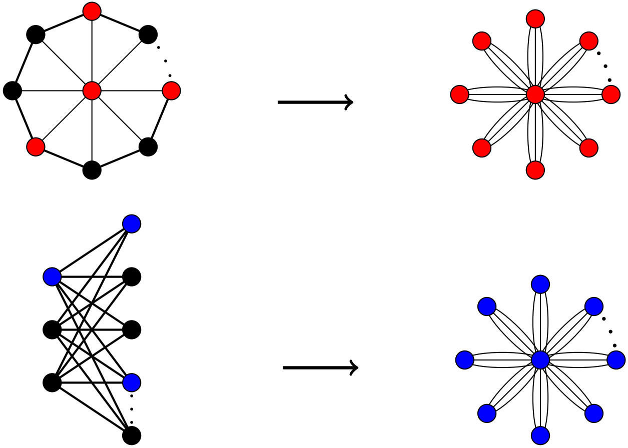

Current Work

Conjecture

There is a function f such that, for every integer $k \geq 4$, every 4-edge-connected graph with order at least $f(k)$ admits an immersed double cycle, $C_{2,k}$.

Current Work

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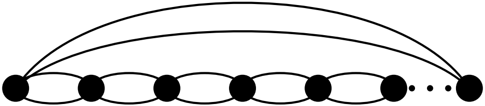

$C_{2,k}$

Proof Sketch

- G is a "large" 4-edge-connected graph
  - G is conn. 1
    - G has a block-tree decomposition
  - G is conn. 2
    - G uniquely decomposes into 2-sums of 3-connected graphs, cycles, and cocycles
  - G is conn. 3
    - G decomposes into wheels, $K_{3,n}$, and I-4-C. graphs
  - G is 4 conn.

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Proof Sketch

Theorem (Oporowski, Oxley, and Thomas)

For every integer $k \geq 4$, there is a function f such that every internally-4-connected graph with at least $f(k)$ vertices contains a topological minor isomorphic to the 2k-spoke double alternating wheel, $K_{4,k}$, the k-rung möbius ladder, the k-rung circular ladder, or $K'_{4,k}$.

Proof Sketch

Theorem (Oporowski, Oxley, and Thomas)

internally-4-connected graph

Thank you!