Quadrangular embeddings
of complete graphs

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Surfaces, embeddings and Euler’s formula

Surfaces:

- $S_h$ orientable, Euler genus $= 2h$

- $N_k$ nonorientable, Euler genus $= k$

Embedding $G \to \Sigma$: draw $G$ without edge crossings.

Euler’s formula: For a surface $\Sigma$ with Euler genus $\gamma$, the Euler characteristic is $\chi(\Sigma) = 2 - \gamma$. For any cellular (nice) graph embedding in $\Sigma$, with $n$ vertices, $m$ edges and $r$ faces, we have

$$n - m + r = \chi(\Sigma) = 2 - \gamma.$$
Colorings of maps

Four Color Conjecture, Francis Guthrie, 1852: The maximum number of colors needed to color a map in the plane, so neighboring countries are different colors, is 4. Easy to show 4 colors necessary, hard to show sufficient. Generally work with dual graph: vertices ↔ faces.

Map Color Conjecture, Heawood, 1890: The maximum number of colors needed for a map on a compact surface with Euler genus $\gamma > 0$ is $H(\gamma) = \frac{7 + \sqrt{1 + 24\gamma}}{2}$. Derived from Euler’s formula and simple assumptions. Easy to show sufficient, hard to show necessary!

$K_7$ on torus, 7 colors needed.
Triangular embeddings of complete graphs

Triangular embedding or triangulation: every face is a triangle bounded by a 3-cycle.

For Map Color Conjecture, examples to show necessity are embeddings of complete graphs that are triangular, or nearly so.

Necessary conditions for triangular embeddings of $K_n$, from Euler’s formula:
   for orientable, $n \equiv 0, 3, 4 \text{ or } 7 \mod 12$;
   for nonorientable, $n \equiv 0, 1, 3 \text{ or } 4 \mod 6$, and $n \geq 5$.

Special case of Map Color Theorem (Ringel, Youngs and others, 1968): Triangular embeddings of $K_n$ exist when the above necessary conditions are satisfied, except that $K_7$ has no nonorientable triangular embedding.

Main tool is algebraic construction: current graphs, due to Gustin, 1963.

Current graph over $\mathbb{Z}_{43}$, embeds $K_{43}$.

So in what ways can we extend this?
Quadrangular embeddings

Quadrangular embedding or quadrangulation: every face is a quadrilateral bounded by a 4-cycle.

Special case of Euler's formula for quadrangulations: $\chi(\Sigma) = n - \frac{1}{2}m$. This must be an integer, and an even integer for an orientable embedding.

Seems natural to consider quadrangular embeddings of complete graphs.

Necessary conditions for quadrangular embeddings of $K_n$, from $\chi(\Sigma) = n - \frac{1}{2}m = n - \frac{1}{2} \binom{n}{2} = \frac{1}{2} n (5 - n)$:

- for orientable, $n \equiv 0$ or $5 \mod 8$;
- for nonorientable, $n \equiv 0$ or $1 \mod 4$, and $n \geq 4$.

Do the embeddings exist?
Quadrangular embeddings of other graphs

Ringel, 1965: $K_{m,n}$ has:

an orientable quadrangular embedding when $(m - 2)(n - 2)$ is divisible by 4 and $m, n \geq 2$;

a nonorientable quadrangular embedding when $mn$ is even and $m, n \geq 3$.

These are minimum genus embeddings.

White, Pisanski and others, 1970 onwards: Existence of quadrangular embeddings for certain cartesian product graphs $G \Box H$. 

$K_{3,4}$ on projective plane  

$K_{3,6}$ on torus
Hartsfield and Ringel’s results

Necessary conditions for quadrangular embeddings of $K_n$, from $\chi(\Sigma) = n - \frac{1}{2}m = n - \frac{1}{2}\binom{n}{2} = \frac{1}{2}n(5-n)$:

for orientable, $n \equiv 0$ or $5 \mod 8$;
for nonorientable, $n \equiv 0$ or $1 \mod 4$, and $n \geq 4$.

Hartsfield & Ringel, 1989: Used current graphs to show $K_n$ has a quadrangular embedding that is
orientable when $n \equiv 5 \mod 8$, and
nonorientable when $n \geq 9$ and $n \equiv 1 \mod 4$ but not when $n = 5$.

What about other cases?
Incomplete and unpublished results

Hartsfield, 1994, claimed: Suppose $G$ is a complete multipartite graph $K_{n_1,n_2,...,n_t}$, with a total of $n$ vertices and $m$ edges. If $t \geq 3$, $n - \frac{1}{2}m$ is an integer, and $G$ is neither $K_5$ nor some graph $K_{1,a,b}$ then $G$ has a nonorientable quadrangular embedding.

In particular, $K_n$ has a nonorientable quadrangular embedding when $n \equiv 0 \mod 4$.

- No general proof for $K_n$ is given, just basis cases ($K_8$ and $K_{12}$) and construction to get from $K_8$ to $K_{16}$, which does not generalize in an obvious way.
- We knew about these claimed results.

Chen, Lawrencenko & Yang, unpublished, written 1998: Determined smallest genus for which $K_n$ has an orientable embedding with all faces of length at least 4, for all $n$.

In particular, $K_n$ has an orientable quadrangular embedding when $n \equiv 0 \mod 8$.

- Proof uses current graphs.
- Apparently withdrawn from journal submission in 1998 in order to combine with some nonorientable results by Hartsfield (those above? or different?).
- Posted on ResearchGate, 2016, after our results posted to arXiv and ResearchGate.
- We were unaware of these results.
Diamond sum of quadrangulations

Bouchet; Magajna, Mohar & Pisanski; Kawarabayashi, Stephens & Zha:
Take two vertices of equal degree in two embeddings. Pair up their neighbours in order around them.

If original embeddings were quadrangulations, so is new embedding.
Nonorientable approach

Induction using two-step diamond sum construction.

Three inputs:

(a) Quadrangular embedding of $K_7^+$ (subdivide $uv$ in $K_7$) on $N_5$.
   Consider $K_7^+$ as $(K_1 \cup K_5) + \overline{K_2}$.

(b) Quadrangular embedding (orientable or nonorientable) of $K_{6,n-1}$: always exists by Ringel’s results.

(c) Quadrangular embedding of $K_n$.

Output:

Nonorientable quadrangular embedding of $K_{n+4}$.
Nonorientable proof

Lemma: If $K_n$ has a quadrangular embedding then $K_{n+4}$ has a nonorientable quadrangular embedding.

Step 1: $K_7^+ \diamond K_{6,n-1} = ((K_1 \cup K_5) + \overline{K_2}) \diamond K_{6,n-1} = (K_1 \cup K_5) + \overline{K_{n-1}}$.

Step 2: $(K_1 \cup K_5) + \overline{K_{n-1}} \diamond K_n = K_{n+4}$.

Theorem: If $n \equiv 0 \text{ or } 1 \mod 4$ and $n \neq 1, 5$ then $K_n$ has a nonorientable quadrangular embedding.

Proof: Apply Lemma starting from $K_4$ on projective plane or from $K_5$ on torus.

Includes new proof of Hartsfield & Ringel’s nonorientable result.
Graphical surfaces

Craft, 1991: Fatten graph into graphical surface $S(G)$. Vertices $\rightarrow$ spheres, edges $\rightarrow$ tubes.

Then get orientable quadrangular embedding on $S(G)$ of composition or lexicographic product $G[\overline{K_2}]$: replace every vertex $v$ of $G$ by two independent vertices $v_N, v_S$, replace every edge by copy of $K_{2,2}$.
Voltages on graphical surface I

Idea: Modify embedding on graphical surface into a voltage graph over group $\mathbb{Z}_2$ so that we can get an embedding of $G[K_4]$.

- Voltage graphs are an algebraic construction, dual to current graphs.
Step 1: Replace all edges by digons, each with one edge of voltage 0, other edge of voltage 1, so voltages alternate around each tube.

- Get two copies of each vertex, so four copies of each vertex of original graph $G$.
- To get quadrangular embedding, need all digons to have total voltage 1, and all faces of degree 4 to have total voltage 0.

Step 1: Replace all edges by digons, each with one edge of voltage 0, other edge of voltage 1, so voltages alternate around each tube.

Step 2: Now assume graph has perfect matching $M$. If $uv \in M$, add vertical digons and loops in quadrilaterals that share a digon.

This will add the edges of a $K_4$ corresponding to each vertex of original graph $G$. 
Voltages on graphical surface IV

Step 1: Replace all edges by digons, each with one edge of voltage 0, other edge of voltage 1, so voltages alternate around each tube.

Step 2: Now assume graph has perfect matching $M$. If $uv \in M$, add vertical digons and loops in quadrilaterals that share a digon.

Step 3: Voltages not quite correct, so swap voltages on one digon for each matching edge $uv$.

Orientable and overall result

**Theorem:** If $G$ has a perfect matching, then $G[K_4]$ has an orientable quadrangular embedding. If $G$ also has a cycle, then $G[K_4]$ has a nonorientable quadrangular embedding as well.

**Corollary:** If $n \equiv 0 \mod 8$ then $K_n$ has an orientable quadrangular embedding.

**Proof:** Let $n = 8k$, and apply the Theorem with $G = K_{2k}$.

- The theorem also gives an alternative proof that when $n \equiv 0 \mod 8$ then $K_n$ has a nonorientable quadrangular embedding.
- By using orientable quadrangular embedding of $K_{11}^+$ (similar to $K_7^+$) can also prove above corollary using diamond sums. Wenzhong Liu has found such an embedding.

**Overall conclusion:** Quadrangular embeddings of $K_n$ exist whenever the necessary conditions are satisfied, except that $K_5$ has no nonorientable quadrangular embedding.
Future work

(1) A minimal quadrangulation is a quadrangular embedding of a simple graph, with no quadrangular embedding of a simple graph of smaller order in the same surface.

- Quadrangular embeddings of complete graphs are minimal. Other examples are known.
- Our theorem on quadrangular embeddings of $G[K_4]$ gives some new results on minimal quadrangulations.

Project: Determine the order of minimal quadrangulations in all surfaces. Wenzhong Liu has made significant progress on this.

(2) Also of interest to look at embeddings of complete graphs in which each face is bounded by a cycle of maximum size, i.e., a hamilton cycle.

- E & Stephens, 2007: For every $n \geq 4$, $K_n$ has a nonorientable embedding with all face boundaries hamilton cycles, except when $n = 5$.
- Necessary condition in orientable case: $n \equiv 2$ or $3 \mod 4$. Embeddings only known to exist for sparse set of values of $n$: E & Stephens, 2008; E & Schroeder, 2013.

Project: Find orientable embeddings of $K_n$ with all face boundaries hamilton cycles whenever the necessary condition holds.

Thank you!