Quadrangular embeddings of complete graphs

Mark Ellingham

Vanderbilt University

Wenzhong Liu

Nanjing University of Aeronautics
and Astronautics, China

Dong Ye and Xiaoya Zha

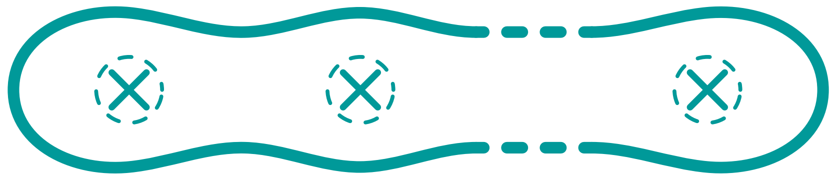
Middle Tennessee State University

Surfaces, embeddings and Euler's formula

Surfaces:

S_h orientable, Euler genus = $2h$

N_k nonorientable, Euler genus = k



Embedding $G \to \Sigma$: draw G without edge crossings.

Euler's formula: For a surface Σ with Euler genus γ, the Euler characteristic is $\chi(\Sigma) = 2 - \gamma$. For any cellular (nice) graph embedding in Σ, with n vertices, m edges and r faces, we have

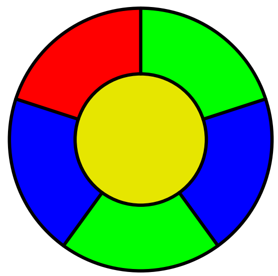
$$n - m + r = \chi(\Sigma) = 2 - \gamma.$$

Colorings of maps

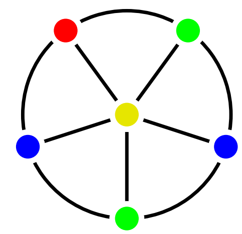
Four Color Conjecture, Francis Guthrie, 1852: The maximum number of colors needed to color a map in the plane, so neighboring countries are different colors, is 4.

Easy to show 4 colors necessary, hard to show sufficient.

Generally work with dual graph: vertices $\leftrightarrow$ faces.



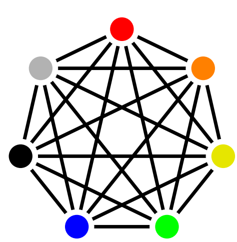
$\leftrightarrow$

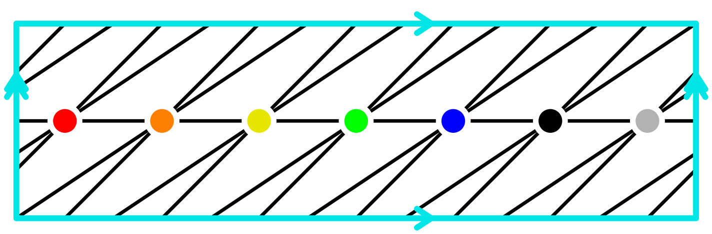


Map Color Conjecture, Heawood, 1890: The maximum number of colors needed for a map on a compact surface with Euler genus $\gamma > 0$ is $H(\gamma) = \dfrac{7 + \sqrt{1 + 24\gamma}}{2}$.

Derived from Euler's formula and simple assumptions.

Easy to show sufficient, hard to show necessary!





K_7 on torus,

7 colors

needed.

Triangular embeddings of complete graphs

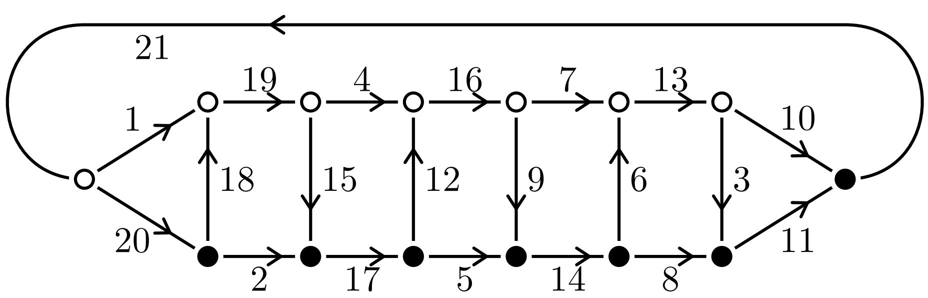
Triangular embedding or triangulation: every face is a triangle bounded by a 3-cycle.

For Map Color Conjecture, examples to show necessity are embeddings of complete graphs that are triangular, or nearly so.

Necessary conditions for triangular embeddings of K_n, from Euler's formula:
for orientable, $n \equiv 0$, 3, 4 or 7 mod 12;
for nonorientable, $n \equiv 0$, 1, 3 or 4 mod 6, and $n \geq 5$.

Special case of Map Color Theorem (Ringel, Youngs and others, 1968): Triangular embeddings of K_n exist when the above necessary conditions are satisfied, except that K_7 has no nonorientable triangular embedding.

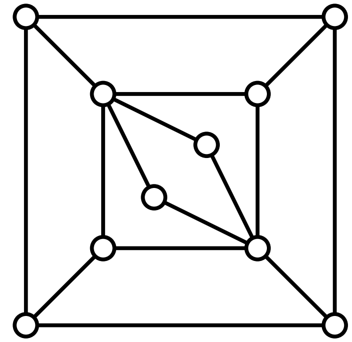
Main tool is algebraic construction: current graphs, due to Gustin, 1963.



Current graph over $\mathbb{Z}_{43}$, embeds K_{43}.

So in what ways can we extend this?

Quadrangular embeddings



Quadrangular embedding or quadrangulation: every face is a quadrilateral bounded by a 4-cycle.

Special case of Euler's formula for quadrangulations: $\chi(\Sigma) = n - \frac{1}{2}m$. This must be an integer, and an even integer for an orientable embedding.

Seems natural to consider quadrangular embeddings of complete graphs.

Necessary conditions for quadrangular embeddings of K_n, from $\chi(\Sigma) = n - \frac{1}{2}m = n - \frac{1}{2}\binom{n}{2} = \frac{1}{2}n(5-n)$:

for orientable, $n \equiv 0$ or 5 mod 8;

for nonorientable, $n \equiv 0$ or 1 mod 4, and $n \geq 4$.

Do the embeddings exist?

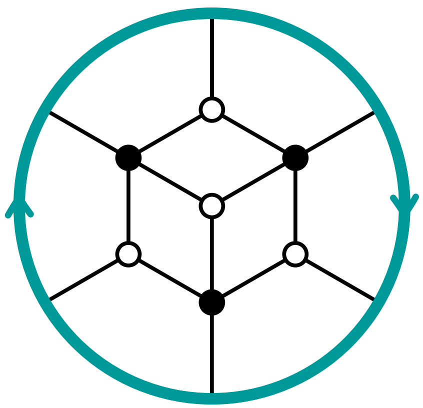
Quadrangular embeddings of other graphs

Ringel, 1965: $K_{m,n}$ has:

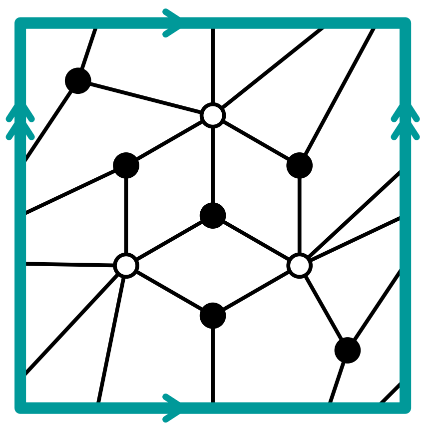
an orientable quadrangular embedding when $(m-2)(n-2)$ is divisible by 4 and $m, n \geq 2$;

a nonorientable quadrangular embedding when mn is even and $m, n \geq 3$.

These are minimum genus embeddings.



$K_{3,4}$ on projective plane



$K_{3,6}$ on torus

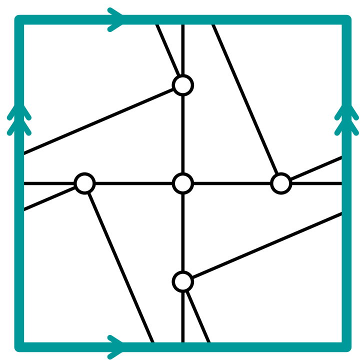
White, Pisanski and others, 1970 onwards: Existence of quadrangular embeddings for certain cartesian product graphs $G \square H$.

Hartsfield and Ringel's results

Necessary conditions for quadrangular embeddings of K_n, from $\chi(\Sigma) = n - \frac{1}{2}m = n - \frac{1}{2}\binom{n}{2} = \frac{1}{2}n(5-n)$:

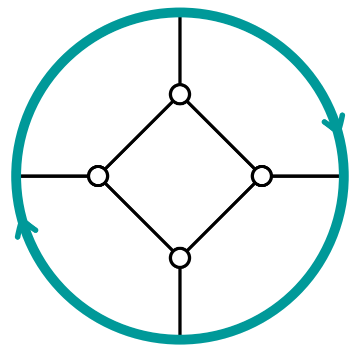
for orientable, $n \equiv 0$ or 5 mod 8;

for nonorientable, $n \equiv 0$ or 1 mod 4, and $n \geq 4$.



K_5 on torus

K_4 on projective plane



Hartsfield & Ringel, 1989: Used current graphs to show K_n has a quadrangular embedding that is

orientable when $n \equiv 5$ mod 8, and

nonorientable when $n \geq 9$ and $n \equiv 1$ mod 4 but not when $n = 5$.

What about other cases?

Incomplete and unpublished results

Hartsfield, 1994, claimed: Suppose G is a complete multipartite graph $K_{n_1,n_2,\ldots,n_t}$, with a total of n vertices and m edges. If $t \geq 3$, $n - \frac{1}{2}m$ is an integer, and G is neither K_5 nor some graph $K_{1,a,b}$ then G has a nonorientable quadrangular embedding.

In particular, K_n has a nonorientable quadrangular embedding when $n \equiv 0 \bmod 4$.

- No general proof for K_n is given, just basis cases (K_8 and K_{12}) and construction to get from K_8 to K_{16}, which does not generalize in an obvious way.
- We knew about these claimed results.

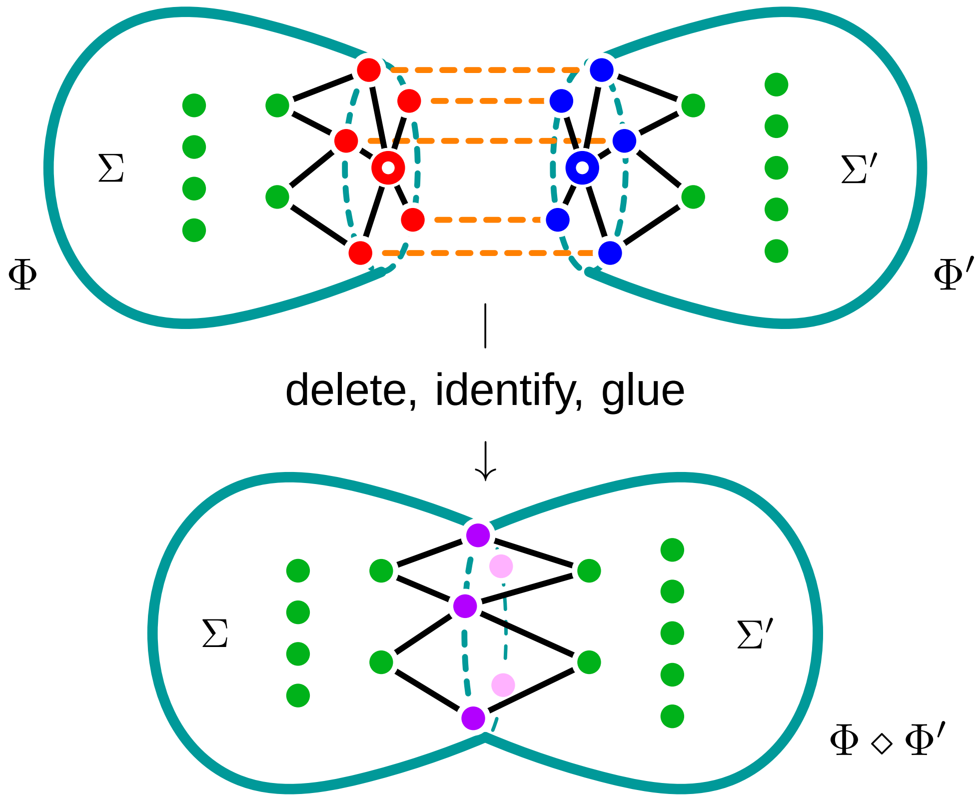
Chen, Lawrencenko & Yang, unpublished, written 1998: Determined smallest genus for which K_n has an orientable embedding with all faces of length at least 4, for all n.

In particular, K_n has an orientable quadrangular embedding when $n \equiv 0 \bmod 8$.

- Proof uses current graphs.
- Apparently withdrawn from journal submission in 1998 in order to combine with some nonorientable results by Hartsfield (those above? or different?).
- Posted on ResearchGate, 2016, after our results posted to arXiv and ResearchGate.
- We were unaware of these results.

Diamond sum of quadrangulations

Bouchet; Magajna, Mohar & Pisanski; Kawarabayashi, Stephens & Zha:
Take two vertices of equal degree in two embeddings. Pair up their neighbours in order around them.



If original embeddings were quadrangulations, so is new embedding.

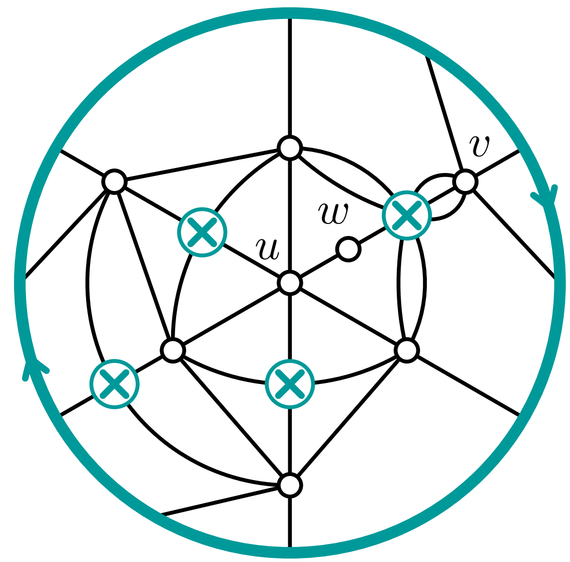
Nonorientable approach

Induction using two-step diamond sum construction.

Three inputs:

(a) Quadrangular embedding of K_7^+ (subdivide uv in K_7) on N_5.

Consider K_7^+ as $(K_1 \cup K_5) + \overline{K_2}$.



(b) Quadrangular embedding (orientable or nonorientable) of $K_{6,n-1}$: always exists by Ringel's results.

(c) Quadrangular embedding of K_n.

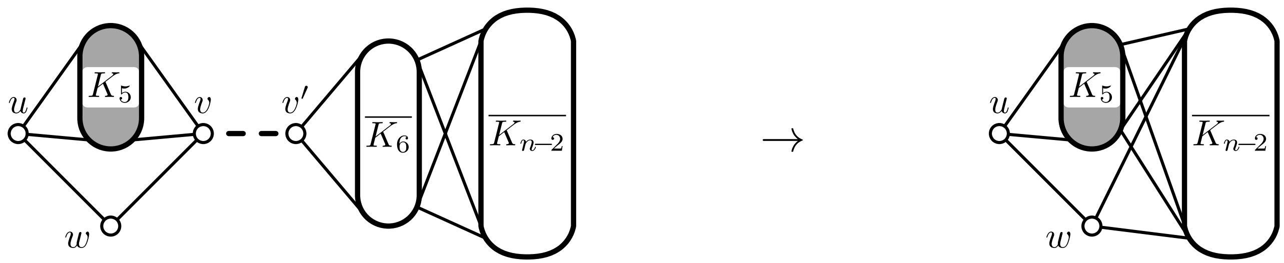
Output:

Nonorientable quadrangular embedding of K_{n+4}.

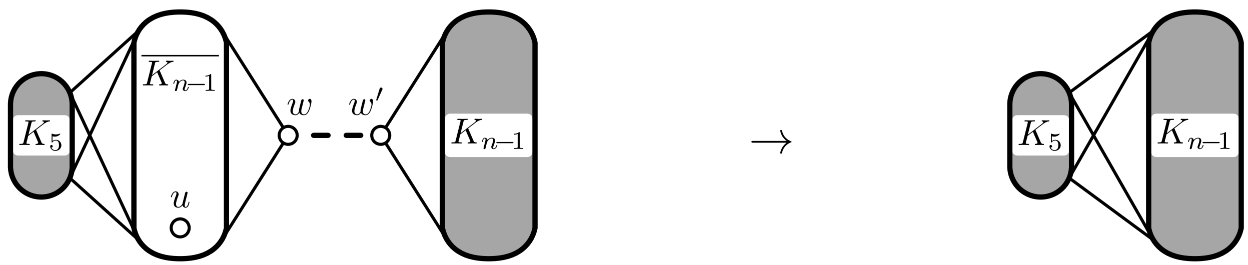
Nonorientable proof

Lemma: If K_n has a quadrangular embedding then K_{n+4} has a nonorientable quadrangular embedding.

Step 1: $K_7^+ \diamond K_{6,n-1} = ((K_1 \cup K_5) + \overline{K_2}) \diamond K_{6,n-1} = (K_1 \cup K_5) + \overline{K_{n-1}}$.



Step 2: $(K_1 \cup K_5) + \overline{K_{n-1}}) \diamond K_n \quad = \quad K_{n+4}$.



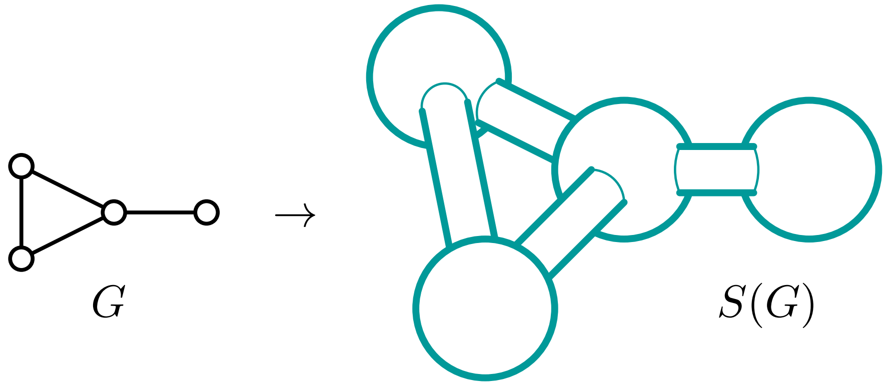
Theorem: If $n \equiv 0$ or 1 mod 4 and $n \neq 1, 5$ then K_n has a nonorientable quadrangular embedding.

Proof: Apply Lemma starting from K_4 on projective plane or from K_5 on torus.

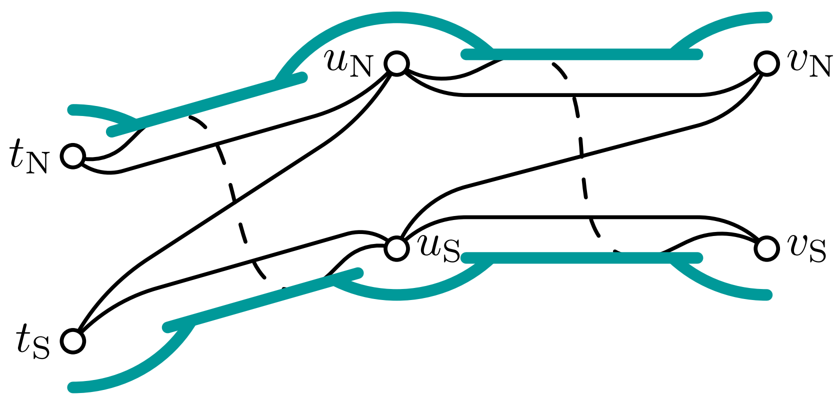
Includes new proof of Hartsfield & Ringel's nonorientable result.

Graphical surfaces

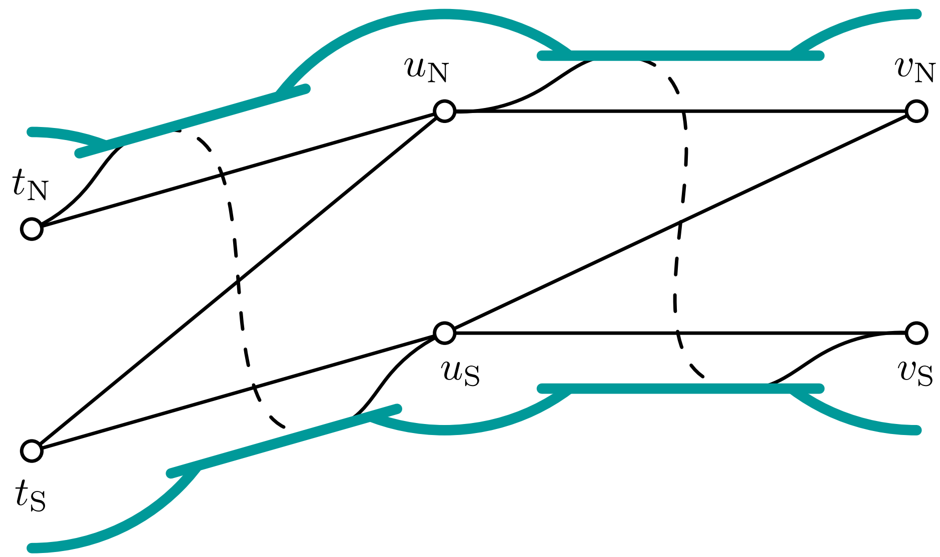
Craft, 1991: Fatten graph into graphical surface $S(G)$. Vertices $\to$ spheres, edges $\to$ tubes.



Then get orientable quadrangular embedding on $S(G)$ of composition or lexicographic product $G[\overline{K_2}]$: replace every vertex v of G by two independent vertices v_{N}, v_{S}, replace every edge by copy of $K_{2,2}$.



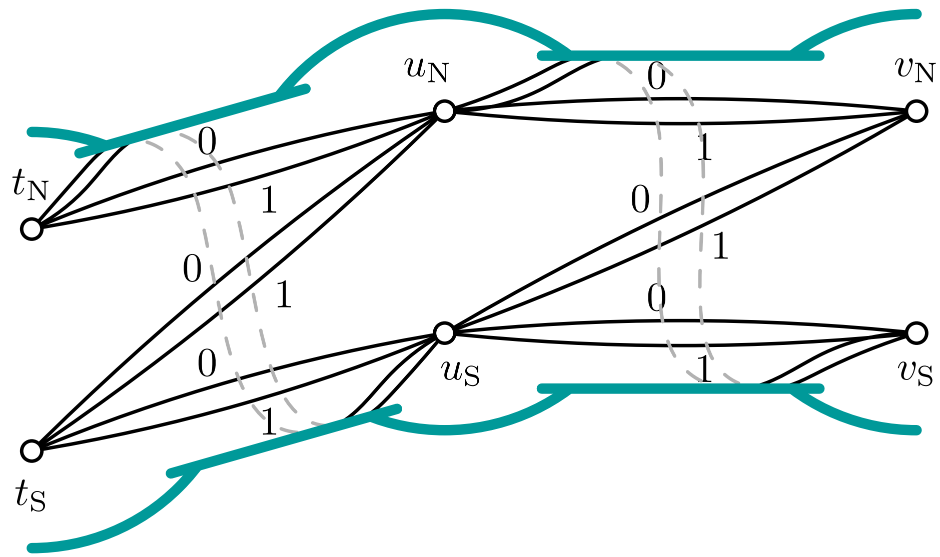
Voltages on graphical surface I



Idea: Modify embedding on graphical surface into a voltage graph over group $\mathbb{Z}_2$ so that we can get an embedding of $G[K_4]$.

- Voltage graphs are an algebraic construction, dual to current graphs.

Voltages on graphical surface II

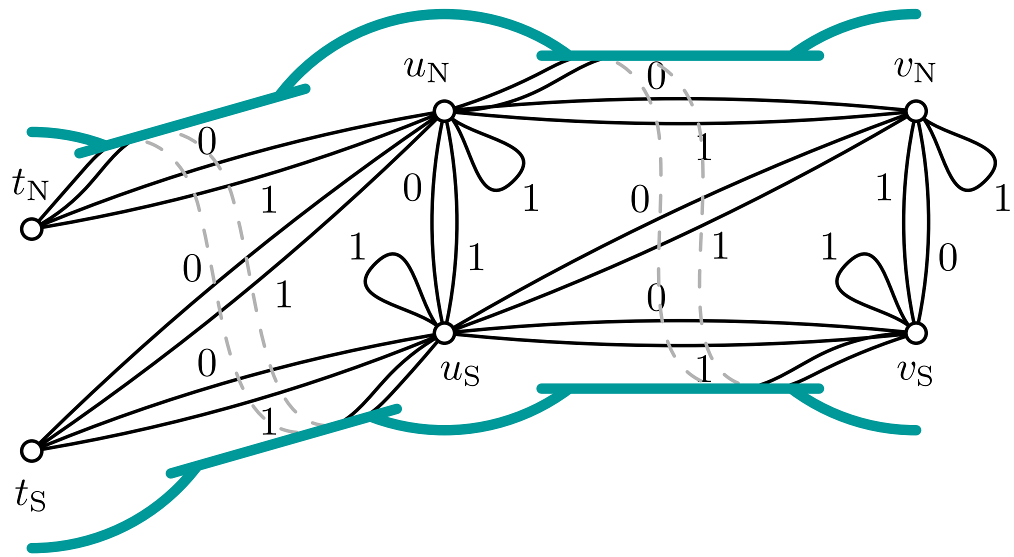


Step 1: Replace all edges by digons, each with one edge of voltage 0, other edge of voltage 1, so voltages alternate around each tube.

- Get two copies of each vertex, so four copies of each vertex of original graph G.
- To get quadrangular embedding, need all digons to have total voltage 1, and all faces of degree 4 to have total voltage 0.

Result: Quadrangular embedding of $G[\overline{K_4}]$.

Voltages on graphical surface III

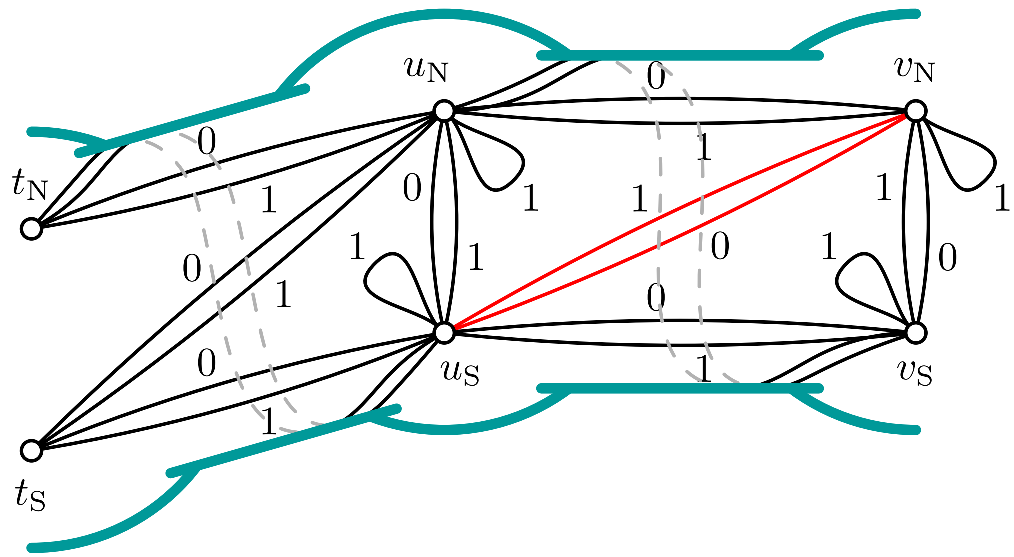


Step 1: Replace all edges by digons, each with one edge of voltage 0, other edge of voltage 1, so voltages alternate around each tube.

Step 2: Now assume graph has perfect matching M. If $uv \in M$, add vertical digons and loops in quadrilaterals that share a digon.

This will add the edges of a K_4 corresponding to each vertex of original graph G.

Voltages on graphical surface IV



Step 1: Replace all edges by digons, each with one edge of voltage 0, other edge of voltage 1, so voltages alternate around each tube.

Step 2: Now assume graph has perfect matching M. If $uv \in M$, add vertical digons and loops in quadrilaterals that share a digon.

Step 3: Voltages not quite correct, so swap voltages on one digon for each matching edge uv.

Result: Quadrangular embedding of $G[K_4]$.

Orientable and overall result

Theorem: If G has a perfect matching, then $G[K_4]$ has an orientable quadrangular embedding. If G also has a cycle, then $G[K_4]$ has a nonorientable quadrangular embedding as well.

Corollary: If $n \equiv 0 \bmod 8$ then K_n has an orientable quadrangular embedding.

Proof: Let $n = 8k$, and apply the Theorem with $G = K_{2k}$.

- The theorem also gives an alternative proof that when $n \equiv 0 \bmod 8$ then K_n has a nonorientable quadrangular embedding.
- By using orientable quadrangular embedding of K_{11}^+ (similar to K_7^+) can also prove above corollary using diamond sums. Wenzhong Liu has found such an embedding.

Overall conclusion: Quadrangular embeddings of K_n exist whenever the necessary conditions are satisfied, except that K_5 has no nonorientable quadrangular embedding.

Future work

(1) A minimal quadrangulation is a quadrangular embedding of a simple graph, with no quadrangular embedding of a simple graph of smaller order in the same surface.

- Quadrangular embeddings of complete graphs are minimal. Other examples are known.
- Our theorem on quadrangular embeddings of $G[K_4]$ gives some new results on minimal quadrangulations.

Project: Determine the order of minimal quadrangulations in all surfaces. Wenzhong Liu has made significant progress on this.

(2) Also of interest to look at embeddings of complete graphs in which each face is bounded by a cycle of maximum size, i.e., a hamilton cycle.

- E & Stephens, 2007: For every $n \geq 4$, K_n has a nonorientable embedding with all face boundaries hamilton cycles, except when $n = 5$.
- Necessary condition in orientable case: $n \equiv 2$ or 3 mod 4. Embeddings only known to exist for sparse set of values of n: E & Stephens, 2008; E & Schroeder, 2013.

Project: Find orientable embeddings of K_n with all face boundaries hamilton cycles whenever the necessary condition holds.

Thank you!