Criticality of Counterexamples to Edge-Hamiltonicity On the Klein Bottle

Joshua Fallon
Louisiana State University

5th Annual Mississippi Discrete Math Workshop

November 4, 2017

The Plane

Theorem (Whitney 1931)

All 4-connected triangulations of the plane are hamiltonian.

The Plane

Theorem (Whitney 1931)

All 4-connected triangulations of the plane are hamiltonian.

Theorem (Tutte 1977)

All 4-connected planar graphs are hamiltonian.

Edge-Hamiltonicity On the Plane

Theorem (Thomassen 1983)

All 4-connected planar graphs are hamilton-connected.

Theorem (Thomas and Yu 1994)

All 4-connected projective planar graphs are hamiltonian.

Like Thomassen's result, the proof of this result in fact proves edge-hamiltonicity of projective planar graphs.

The Torus - Partial Results (Tilings)

Theorem (Altshuler 1972)

All 6-connected toroidal graphs are hamiltonian.

These graphs are 6-regular triangulations with a grid structure.

The Torus - Partial Results (Tilings)

Theorem (Altshuler 1972)

All 6-connected toroidal graphs are hamiltonian.

These graphs are 6-regular triangulations with a grid structure.

Theorem (Altshuler 1972)

All 4-connected toroidal quadrangulations are hamiltonian.

These graphs are 4-regular quadrangulations with a grid structure.

The Torus - Partial Results (5-connected)

Theorem (Brunet and Richter 1995)

All 5-connected triangulations of the torus are hamiltonian.

The Torus - Partial Results (5-connected)

Theorem (Brunet and Richter 1995)

All 5-connected triangulations of the torus are hamiltonian.

Theorem (Thomas and Yu 1997)

All 5-connected toroidal graphs are edge-hamiltonian.

The Klein Bottle

Theorem (Brunet, Nakamoto, and Negami 1998)

Every 5-connected triangulation of the Klein bottle is hamiltonian.

The Klein Bottle

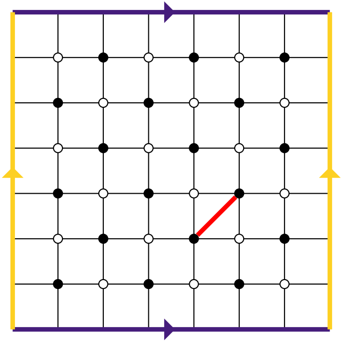
Theorem (Brunet, Nakamoto, and Negami 1998)

Every 5-connected triangulation of the Klein bottle is hamiltonian.

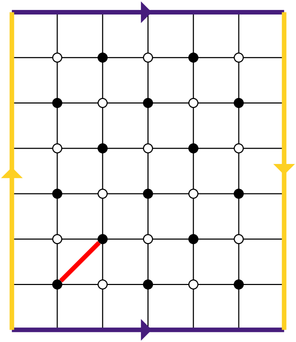
Conjecture (Nash-Williams 1973)

Every 4-connected Klein bottle graph is hamiltonian.

Counterexamples to Edge-Hamiltonicity



Counterexamples to Edge-Hamiltonicity

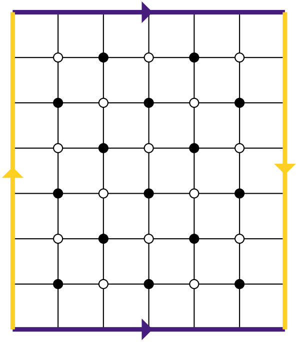


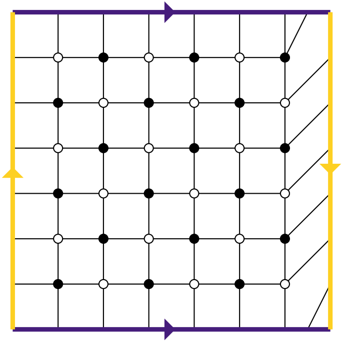
The Torus

Theorem (Ellingham and Marshall 2013)

Let G be a 4-connected, 4-regular bipartite simple graph on the torus with partition sets of white and black vertices. If we add a nonempty set E_1 of one or more black-black diagonals to G, then no element of E_1 lies on a hamilton cycle in $G \cup E_1$. However, if we add one further white-white diagonal e_2 in a quadrangle of $G \cup E_1$, then each edge of $G \cup E_1 \cup \{e_2\}$ lies on a hamilton cycle of that graph.

Bipartite Grid-Type Quadrangulations



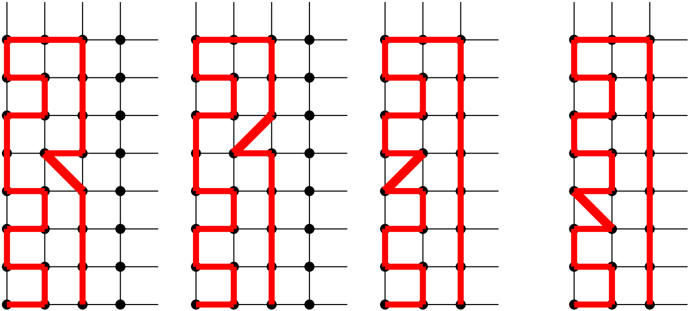


Grid-Type Quadrangulations

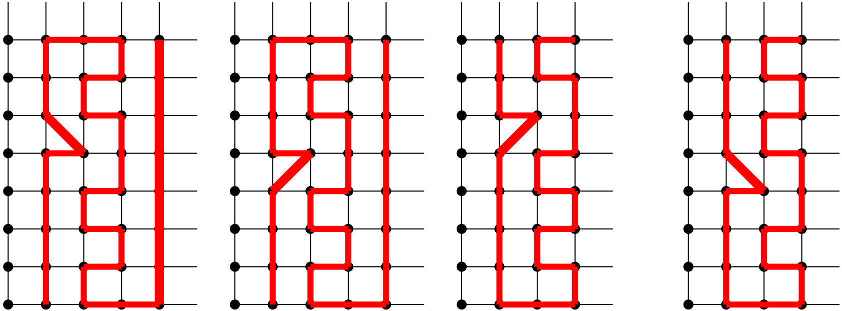
Theorem (F.)

Let G be a bipartite grid-type quadrangulation of the Klein bottle with partition sets of white and black vertices. If we add a nonempty set E_1 of one or more black-black diagonals to the quadrangles of G, then no element of E_1 lies on a hamilton cycle in $G \cup E_1$. However, if we add one further white-white diagonal e_2 in a quadrangle of $G \cup E_1$, then each edge of $G \cup E_1 \cup \{e_2\}$ lies on a hamilton cycle of that graph.

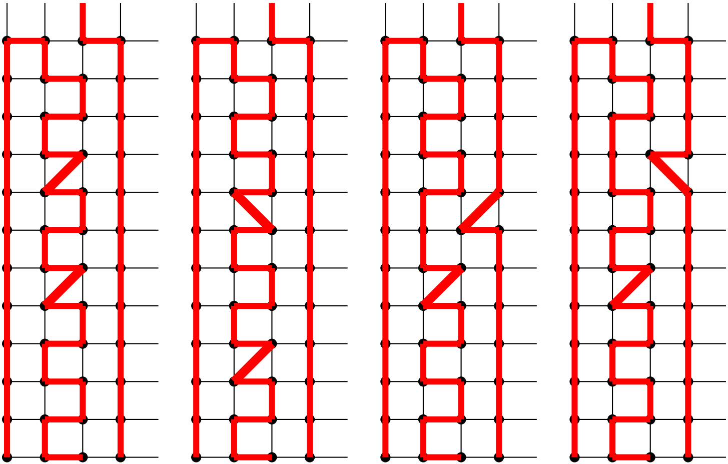
Black-Black Diagonals



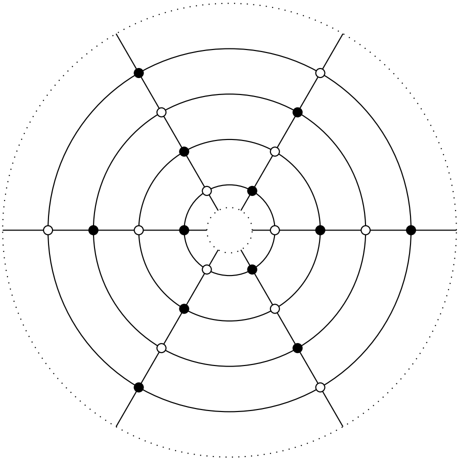
White-White Diagonals

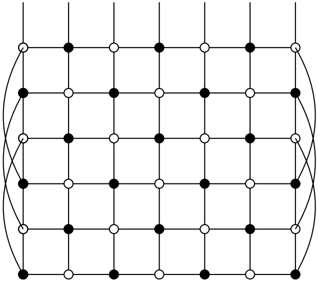


Nearby Diagonals



Bipartite Ladder-Type Quadrangulations



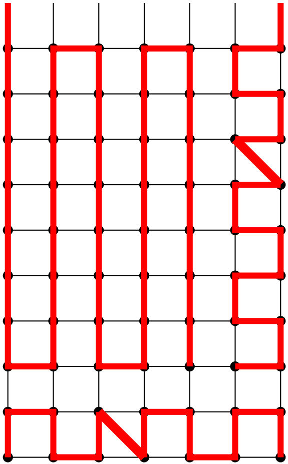


Ladder-Type Quadrangulations

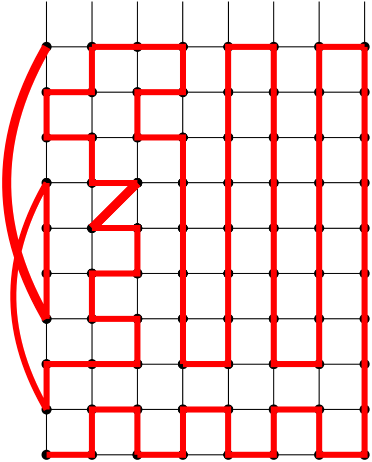
Theorem (F.)

Let G be a bipartite ladder-type quadrangulation of the Klein bottle with partition sets of white and black vertices. If we add a nonempty set E_1 of one or more black-black diagonals to the quadrangles of G, then no element of E_1 lies on a hamilton cycle in $G \cup E_1$. However, if we add one further white-white diagonal e_2 in a quadrangle of $G \cup E_1$, then each edge of $G \cup E_1 \cup \{e_2\}$ lies on a hamilton cycle of that graph.

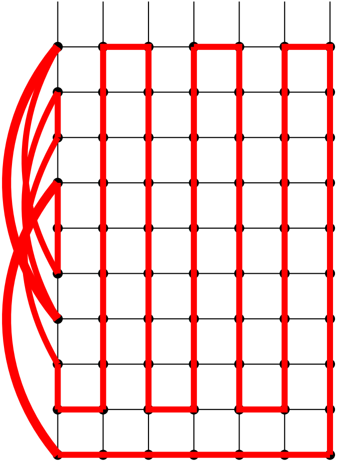
Both Diagonals In the Grid



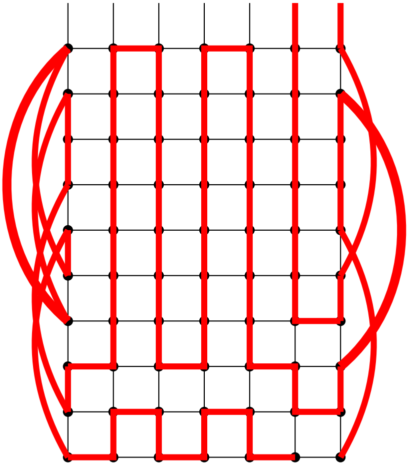
One Diagonal In the Grid



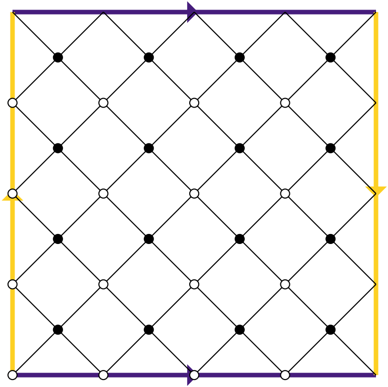
Diagonals In the Same Möbius Ladder



Diagonals In Opposite Möbius Ladders



Mesh-Type Quadrangulations



Future Work

Conjecture

Let G be a 4-connected, 4-regular, bipartite simple graph on the Klein bottle with partition sets of white and black vertices. If we add a nonempty set E_1 of one or more black-black diagonals to G, then no element of E_1 lies on a hamilton cycle in $G \cup E_1$. However, if we add one further white-white diagonal e_2 in a quadrangle of $G \cup E_1$, then each edge of $G \cup E_1 \cup \{e_2\}$ lies on a hamilton cycle of that graph.

Thank you!