Criticality of Counterexamples to Edge-Hamiltonicity On the Klein Bottle

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The Plane

Theorem (Whitney 1931)

All 4-connected triangulations of the plane are hamiltonian.
The Plane

Theorem (Whitney 1931)
All 4-connected triangulations of the plane are hamiltonian.

Theorem (Tutte 1977)
All 4-connected planar graphs are hamiltonian.
Theorem (Thomassen 1983)

All 4-connected planar graphs are hamilton-connected.
The Projective Plane

Theorem (Thomas and Yu 1994)

All 4-connected projective planar graphs are hamiltonian.

Like Thomassen’s result, the proof of this result in fact proves edge-hamiltonicity of projective planar graphs.
Theorem (Altshuler 1972)

All 6-connected toroidal graphs are hamiltonian.

These graphs are 6-regular triangulations with a grid structure.
Theorem (Altshuler 1972)

All 6-connected toroidal graphs are hamiltonian.

These graphs are 6-regular triangulations with a grid structure.

Theorem (Altshuler 1972)

All 4-connected toroidal quadrangulations are hamiltonian.

These graphs are 4-regular quadrangulations with a grid structure.
Theorem (Brunet and Richter 1995)

All 5-connected triangulations of the torus are hamiltonian.
### Theorem (Brunet and Richter 1995)

All 5-connected triangulations of the torus are hamiltonian.

### Theorem (Thomas and Yu 1997)

All 5-connected toroidal graphs are edge-hamiltonian.
The Klein Bottle

Theorem (Brunet, Nakamoto, and Negami 1998)

Every 5-connected triangulation of the Klein bottle is hamiltonian.
The Klein Bottle

Theorem (Brunet, Nakamoto, and Negami 1998)
Every 5-connected triangulation of the Klein bottle is hamiltonian.

Conjecture (Nash-Williams 1973)
Every 4-connected Klein bottle graph is hamiltonian.
Counterexamples to Edge-Hamiltonicity
Counterexamples to Edge-Hamiltonicity

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Klein Bottle Edge-Hamiltonicity
Theorem (Ellingham and Marshall 2013)

Let $G$ be a 4-connected, 4-regular bipartite simple graph on the torus with partition sets of white and black vertices. If we add a nonempty set $E_1$ of one or more black-black diagonals to $G$, then no element of $E_1$ lies on a hamilton cycle in $G \cup E_1$. However, if we add one further white-white diagonal $e_2$ in a quadrangle of $G \cup E_1$, then each edge of $G \cup E_1 \cup \{e_2\}$ lies on a hamilton cycle of that graph.
Theorem (F.)

Let $G$ be a bipartite grid-type quadrangulation of the Klein bottle with partition sets of white and black vertices. If we add a nonempty set $E_1$ of one or more black-black diagonals to the quadrangles of $G$, then no element of $E_1$ lies on a hamilton cycle in $G \cup E_1$. However, if we add one further white-white diagonal $e_2$ in a quadrangle of $G \cup E_1$, then each edge of $G \cup E_1 \cup \{e_2\}$ lies on a hamilton cycle of that graph.
Black-Black Diagonals
Nearby Diagonals
Bipartite Ladder-Type Quadrangulations

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Klein Bottle Edge-Hamiltonicity
Theorem (F.)

Let $G$ be a bipartite ladder-type quadrangulation of the Klein bottle with partition sets of white and black vertices. If we add a nonempty set $E_1$ of one or more black-black diagonals to the quadrangles of $G$, then no element of $E_1$ lies on a hamilton cycle in $G \cup E_1$. However, if we add one further white-white diagonal $e_2$ in a quadrangle of $G \cup E_1$, then each edge of $G \cup E_1 \cup \{e_2\}$ lies on a hamilton cycle of that graph.
Both Diagonals In the Grid

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Klein Bottle Edge-Hamiltonicity
One Diagonal In the Grid
Diagonals In the Same Möbius Ladder
Diagonals In Opposite Möbius Ladders

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Klein Bottle Edge-Hamiltonicity
Mesh-Type Quadrangulations
Conjecture

Let $G$ be a 4-connected, 4-regular, bipartite simple graph on the Klein bottle with partition sets of white and black vertices. If we add a nonempty set $E_1$ of one or more black-black diagonals to $G$, then no element of $E_1$ lies on a hamilton cycle in $G \cup E_1$. However, if we add one further white-white diagonal $e_2$ in a quadrangle of $G \cup E_1$, then each edge of $G \cup E_1 \cup \{e_2\}$ lies on a hamilton cycle of that graph.
Thank you!