Laminar Matroids

And Generalized Laminar Matroids

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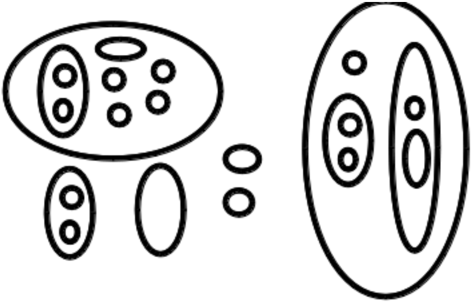
Laminar Family



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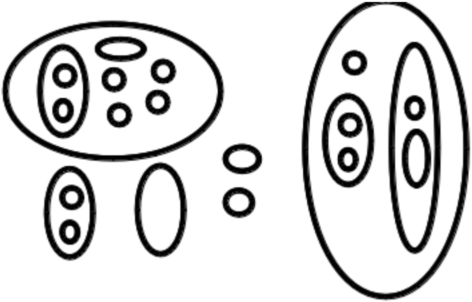

A family $\mathscr{A}$ of sets is laminar if, for all $A_1, A_2 \in \mathscr{A}$, either $A_1 \cap A_2 = \emptyset$, or $A_i \subseteq A_j$, for distinct $i, j \in \{1, 2\}$.

What is a Laminar Matroid?

E a finite set

$\mathscr{A}$ a laminar family of subsets of E

c a *capacity* function from $\mathscr{A}$ to the non-negative integers

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c a *capacity* function from $\mathscr{A}$ to the non-negative integers

I is **independent** in $M(E, \mathscr{A}, c)$ if, for all $A \in \mathscr{A}$,

$$|I \cap A| \leq c(A).$$

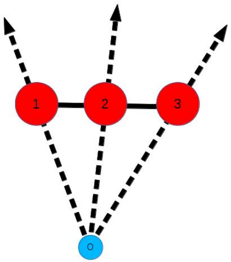
Geometric Presentation

The following are dependent sets.

- Two dots on a point.

The following are dependent sets.

- Two dots on a point.
- Three dots on a line.



Geometric Presentation

The following are dependent sets.

- Two dots on a point.
- Three dots on a line.
- Four dots on a plane.

Geometric Presentation

The following are dependent sets.

- Two dots on a point.
- Three dots on a line.
- Four dots on a plane.
- Five dots in space.

What is a Minor?

Delete e: Remove e.

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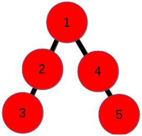
Delete 1



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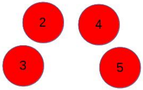

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Contract e: Project from e onto a hyperplane that does not contain e.

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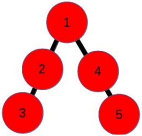

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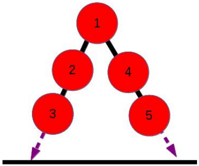

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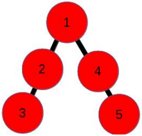

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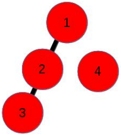

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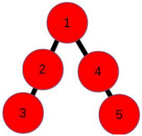

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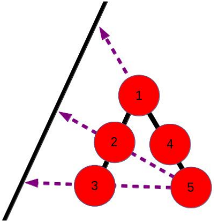

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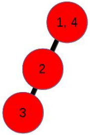
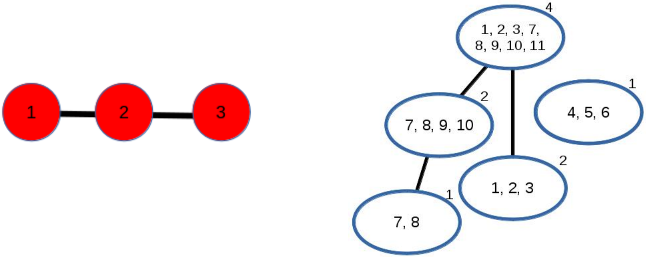


Matroids

A *matroid* is a nice notion of independence and dependence in a finite set.

Geometric Representation of a Laminar Matroid



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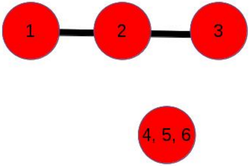

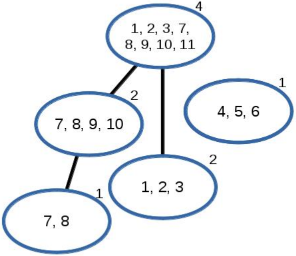

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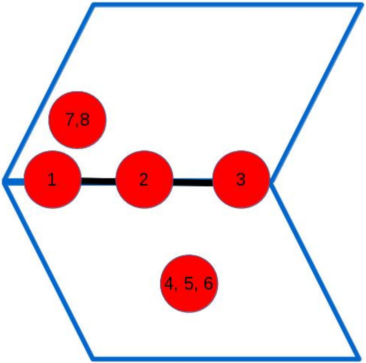

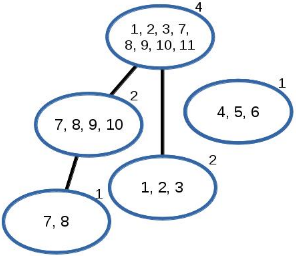

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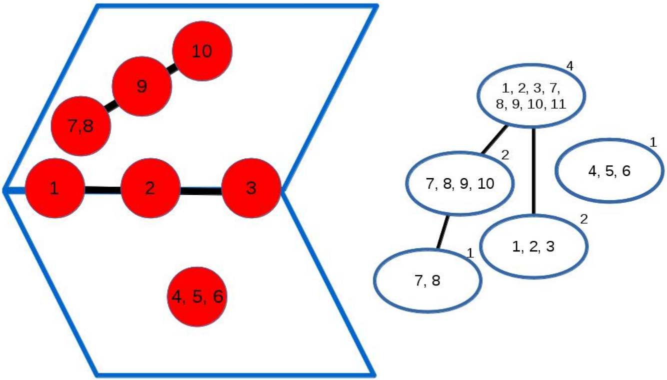

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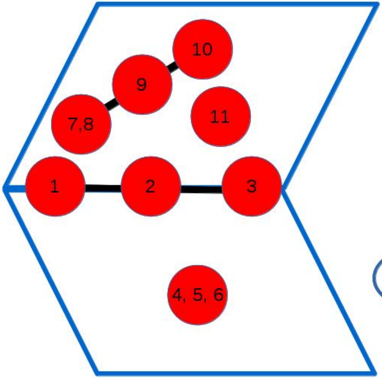
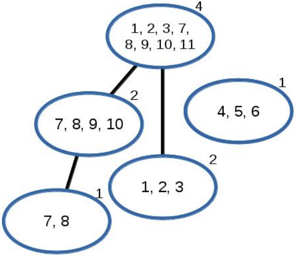
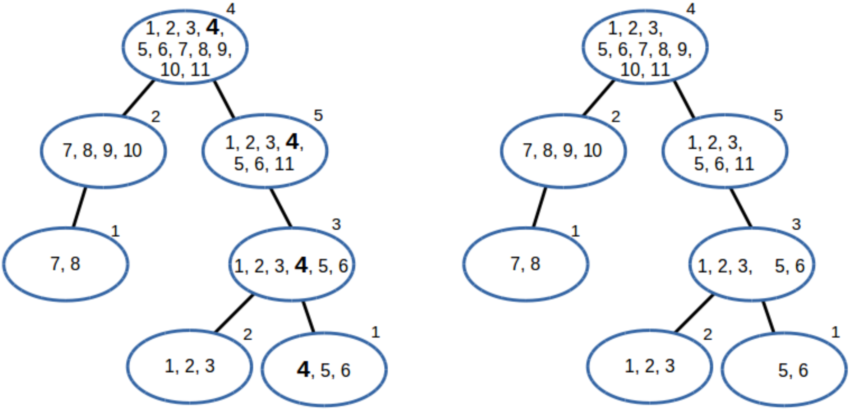




Minors of Laminar Matroids



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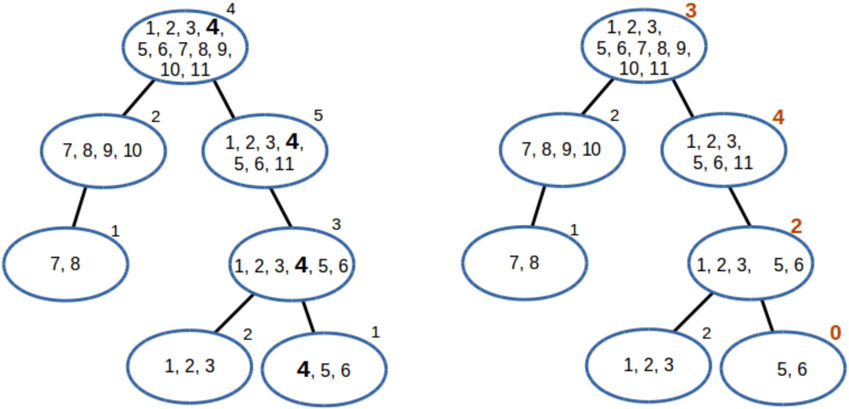
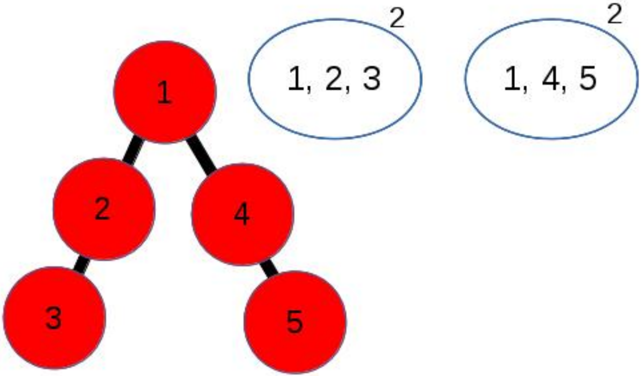
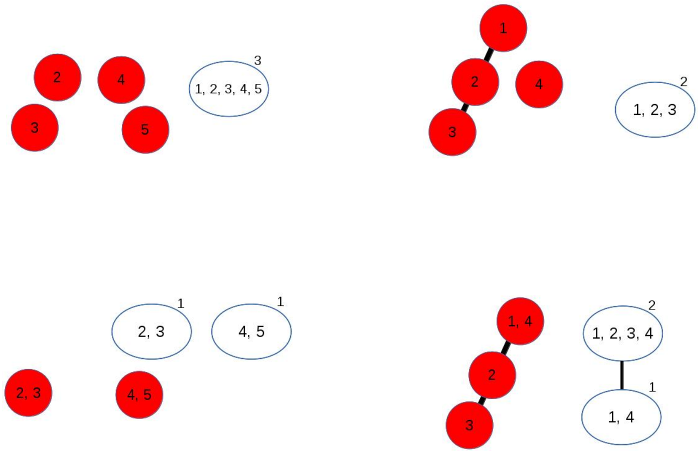


Not Laminar



An Excluded Minor

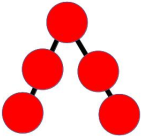


The Excluded Minors

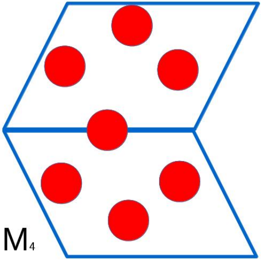
Theorem

The excluded minors of laminar matroids are:

M_3



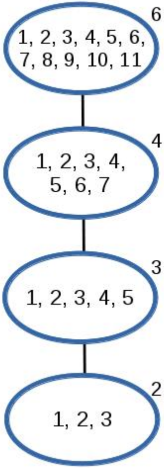
M_4



. . .

Nested Matroids

These are laminar matroids which have a representation where the family $\mathscr{A}$ looks like a path.



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Theorem (O., Prendergast, and Row)

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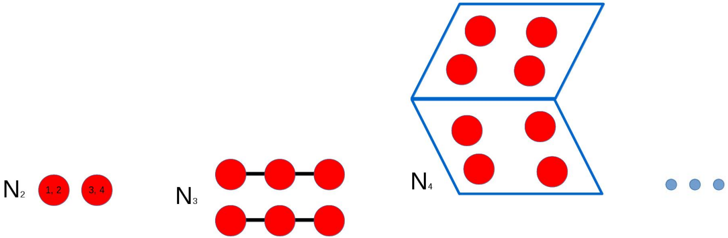
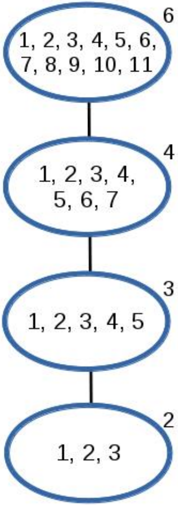


Another Look at Circuits



A circuit is a minimal dependent set.

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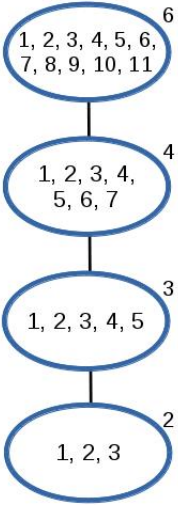

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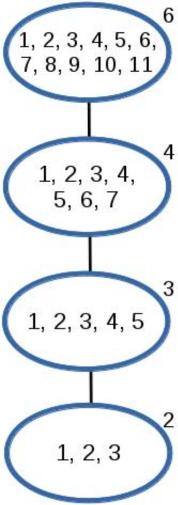

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$\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$

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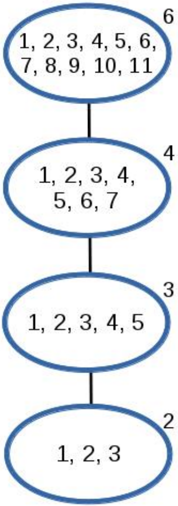

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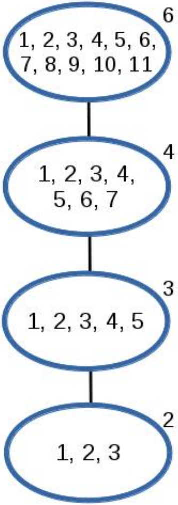

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etc.

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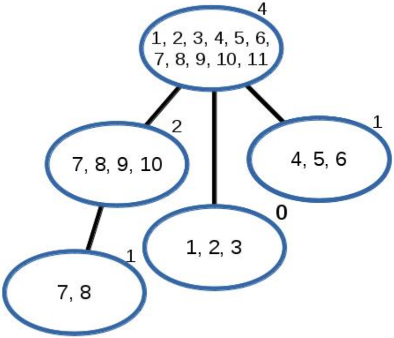
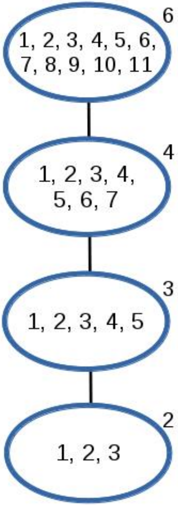


A Circuit is a minimally dependent set.

Lemma

For a laminar matroid $M(E, \mathscr{A}, c)$, a set C is a circuit if it is a minimal set such that $C \subseteq A$ and $|C| = c(A) + 1$ for some $A \in \mathscr{A}$.

Hamiltonian Flats



If $X \subseteq E$, we define $cl(X)$, the closure of X, to be

$X \cup \{e : \text{there is a circuit } C \text{ with } e \in C \subseteq e \cup X\}$.

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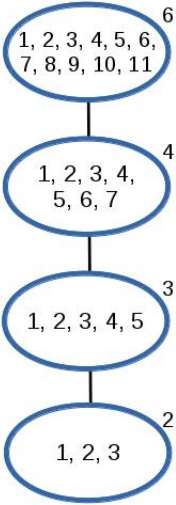

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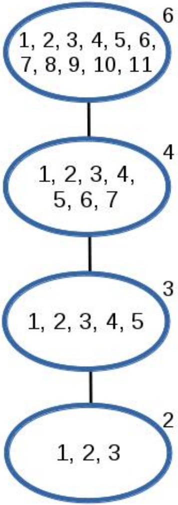

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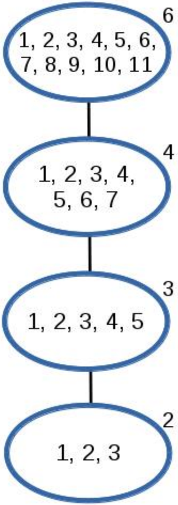
Our Hamiltonian flats are:



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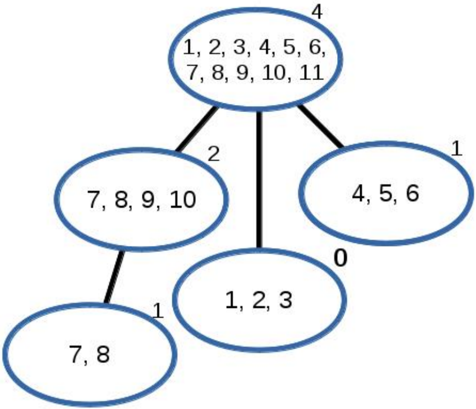
Hamiltonian Flats



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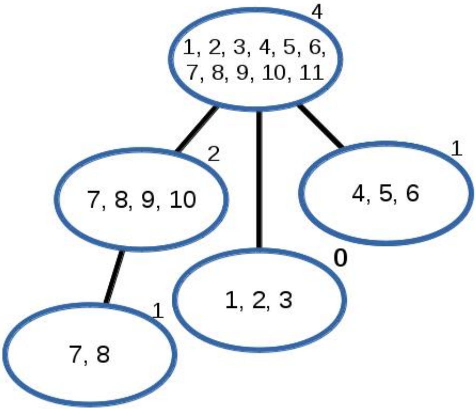

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Hamiltonian Flats

Theorem

A matroid is nested if and only if its Hamiltonian flats form a chain under inclusion.

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Theorem

A matroid M is laminar if and only if for every independent set X of size 1, the Hamiltonian flats of M containing X form a chain under inclusion.

A Generalization

Let $\mathscr{M}_k$ (i.e. k-closure-laminar) be the class of all matroids such that for every independent set X of size k, the Hamiltonian flats of M containing X form a chain under inclusion.

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- $\mathscr{M}_k$ is not minor-closed for any $k \geq 4$.

Another Generalization

Theorem

A matroid is nested if and only if for every two circuits C_1 and C_2 either $cl(C_1) \subseteq cl(C_2)$ or $cl(C_2) \subseteq cl(C_1)$.

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Let $\mathscr{N}_k$ (i.e. k-laminar) be the class of matroids such that for any two circuts C_1 and C_2 with $|C_1 \cap C_2 \geq k$ either $cl(C_1) \subseteq cl(C_2)$ or $cl(C_2) \subseteq cl(C_1)$.

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- $\mathscr{M}_k \subsetneq \mathscr{N}_k$ for all k.

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- $\mathscr{N}_k$ is minor-closed for all k.
- $\mathscr{M}_k \subsetneqq \mathscr{N}_k$ for all k.
- The excluded minors for $\mathscr{N}_k$ are known.

Thank You

Thank You.