Characterization and enumeration of 3-regular permutation graphs

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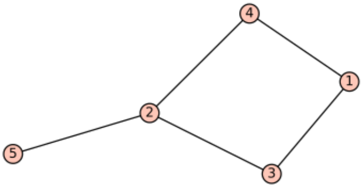
A permutation graph is:

- A *n*-vertex graph representing a permutation on *n* letters, where each vertex corresponds to a letter, and vertices are adjacent if and only if the corresponding letters are inverted in the permutation.

Example

Let $\pi = [3, 4, 1, 5, 2]$.



A permutation graph is:

- A circle graph with an equator.

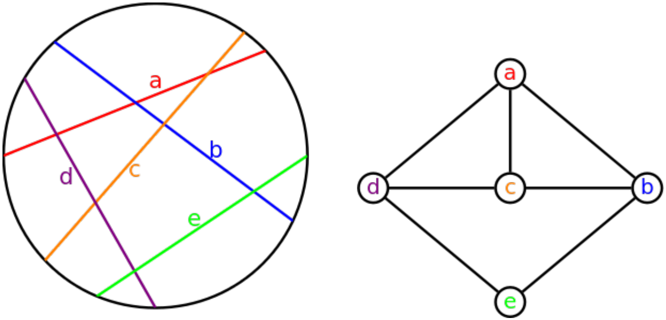


Image: Wikipedia

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- Two copies of a graph joined by a perfect matching (eg. the Petersen graph).

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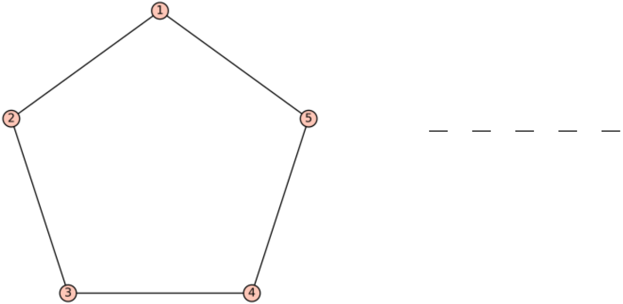

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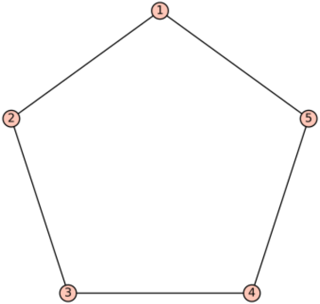

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Where can 1 and 5 go?

Definitions

- Given a permutation $\pi = [\pi(1), \pi(2), \ldots, \pi(n)]$, we define an *inversion* to be a pair $(\pi(i), \pi(j))$ when $i < j$ and $\pi(i) > \pi(j)$.

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- If G is a permutation graph corresponding to permutation π, then π is a *realizer* of G.

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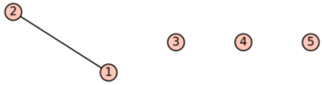
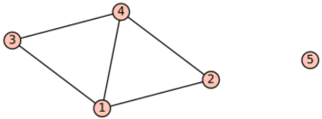


$(1, 2)$, or $[2, 1, 3, 4, 5]$



$(1, 4)$, or $[4, 2, 3, 1, 5]$

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- The vertex connectivity the minimal number of letters that must be removed to achieve this.
- Pattern avoidance in permutations corresponds precisely to avoiding the induced subgraphs that the patterns realizes. The realizers of P_2 are $[3, 1, 2]$ and $[2, 3, 1]$. Thus a permutation graph realized by a $\{[3, 1, 2], [2, 3, 1]\}$-avoiding permutation is precisely a graph without P_2 as an induced subgraph.

Blow-ups

Let G and H be permutation graphs. We can construct a new permutation graph by taking their disjoint union, deleting a vertex v of G, and adding an edge between every vertex in H and every vertex formerly in $N(V)$. We call this a *blow-up* if H is K_n or I_n.

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A path P_2 on 2 vertices can be blown up into $K_{3,3}$ by replacing every vertex with I_3. P_2 is realized by $[2,1]$. I_3 is realized by $[1,2,3]$, and $K_{3,3}$ is realized by $[4,5,6,1,2,4]$.

Infinitely many r-regular permutation graphs

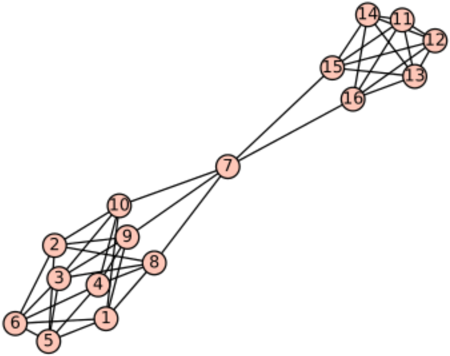
Theorem

For every $r > 2$, there are infinitely many connected r-regular permutation graphs.

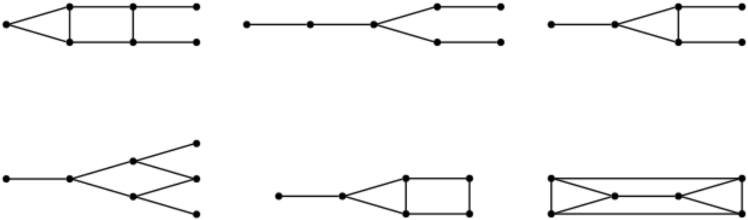
By blowing up paths, we can construct infinitely many r-regular permutation graphs for $r \geq 3$.
Start with a path of of length $2 \pmod 4$, blow up the beginning to K_2 and the end to K_{r-1}, and blow up the intermediate sequences of four vertices into $I_{r-1}, I_{r-2}, I_1, I_2$.

A 5-regular blow-up of P_6

Example



Forbidden induced subgraphs



And cycles of length 5 or greater.

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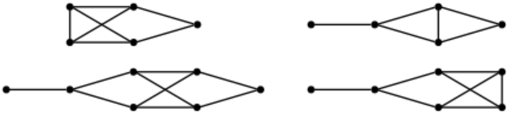

Induced subgraphs of boxcar graphs

Lemmas

- Every 3-regular permutation graph that is the blow-up of a path is isomorphic to K_4, $K_{3,3}$, or a boxcar graph.

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- If G^* is a graph with maximum degree d, and if G is a graph of minimum order such that G^* is a blow-up of G, then G has no degree d twins.
- A 3-regular permutation graph cannot have a ladder with 4 or more rungs as a subgraph.

Enumeration

Question: How many connected 3-regular permutation graphs are there on n vertices?

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Together the "cabooses" of a boxcar graph contribute 10 vertices. Each other car contributes either 4 vertices or 6 vertices. The question can be reduced to counting integer compositions into parts of size 4 and 6 (equivalently, parts of size 2 and 3). We also have to count a composition and its reverse as being the same.

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Call such equivalence classes of composition of n into 2s and 3s *sequences for* n.

A recursion relation

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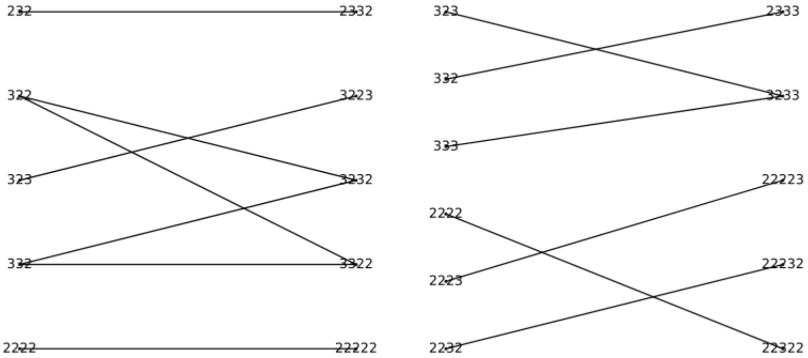
Insert the missing symbol in the center, to either side of the center if the length of the sequence is odd.

Overcounting only happens when n is odd



Enumeration theorem

Theorem

Let $a(n)$ be the number of connected 3-regular permutation graphs on n vertices. If n is an odd integer or if $n \in \{2, 8, 12\}$, then $a(n) = 0$. If $n \in \{4, 6, 10, 14, 16, 18, 20\}$, then $a(n) = 1$. For even $n > 20$, we have

$$a(n) = \begin{cases} a(n-4) + a(n-6) & \text{if } n \equiv 2 \pmod 4 \\ a(n-4) + a(n-6) - t(\frac{n-20}{4}) & \text{if } n \equiv 0 \pmod 4, \end{cases}$$

where $t(x)$ is the number of compositions of x into parts of size 2 or 3.

Thank you!