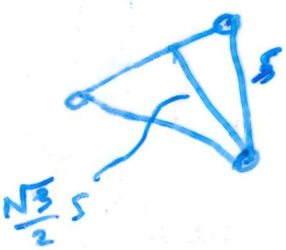
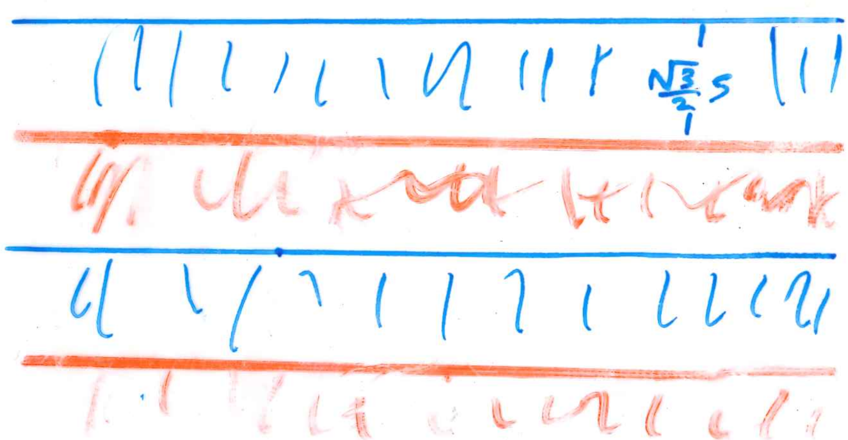
Some Wrong-Way Euclidean Coloring Problems [They start in $\mathbb{R}^n$, $n > 1$, and wind up in $\mathbb{Z} = \{\text{integers}\}$.]

Euclidean coloring problem from 1973–1976 — still open: For which $T \subseteq \mathbb{R}^2$, $|T| = 3$, can $\mathbb{R}^2$ be 2-colored so that no T' congruent to T is monochromatic?

[Congruent? T' is congruent to T $\Leftrightarrow$ can go from T to T' by a rotation and a translation.]

Known by 1976: If $T = \{\text{vertices of an equilateral triangle}\}$ then yes:





Tile the plane with half-open strips of width $\frac{\sqrt{3}}{2}s$; color them alternately orange, blue, orange, blue, ...

Jargon from that era: equilateral triangles are "not Ramsey" in $\mathbb{R}^2$.

Conjecture from that era (Erdős, Graham, Montgomery, Rothschild, Spencer, & Straus): Only equilateral triangles are un Ramsey in $\mathbb{R}^2$.

Known by 1985(?) (Leslie Shader):
Loads of 3-sets, including the vertices of any right triangle, are Ramsey in $\mathbb{R}^2$ (= no monochromatic-congruent-copy-forbidding 2 coloring of $\mathbb{R}^2$).

After 1985: **SILENCE**

Digression: Hasn't been much work on

$\chi_c(\mathbb{R}^2, T)$ = smallest no. of colors needed to color $\mathbb{R}^2$ so that no T' congruent to T is monochromatic
$(|T| \geq 2)$

Clearly $\chi_c(\mathbb{R}^2, T) \leq \chi(\mathbb{R}^2, 1) \in \{4, 5, 6, 7\}$

$\qquad\qquad\qquad\qquad\qquad\quad \chi_c(\mathbb{R}^2, \{u, v\})$, $u, v \in \mathbb{R}^2$, $|u-v| = 1$.

PJ's conjecture: $|T| \geq 3 \Rightarrow \chi_c(\mathbb{R}^2, T) \leq 4$.

Problem: Does there exist a positive integer k s.t. $\chi_c(\mathbb{R}^2, F) = 2$ for all $F \subseteq \mathbb{R}^2$, $|F| \geq k$?

© 2000: PJ casts about for suitable REU problems. What about coloring Euclidean spaces so as to forbid monochromatic translates of given figures?

Difficult to ~~find~~ such problems that aren't trivial.

[However: here's an EGMRSS result from 1976: If $\mathbb{R}^2$ is colored with orange and blue so that some distance is forbidden for blue, then the orange set contains a translate of every 3-set in $\mathbb{R}^2$.]

PJ's hope for a non-trivial forbidding-monochromatic-translates question:
Is it true that for any $T_1, T_2 \subseteq \mathbb{R}^2$, $|T_1| = |T_2| = 3$, there is a 2-coloring of $\mathbb{R}^2$ that forbids monochromatic translates of each T_i? [I.e., no translate of either T_i is monochromatic.]

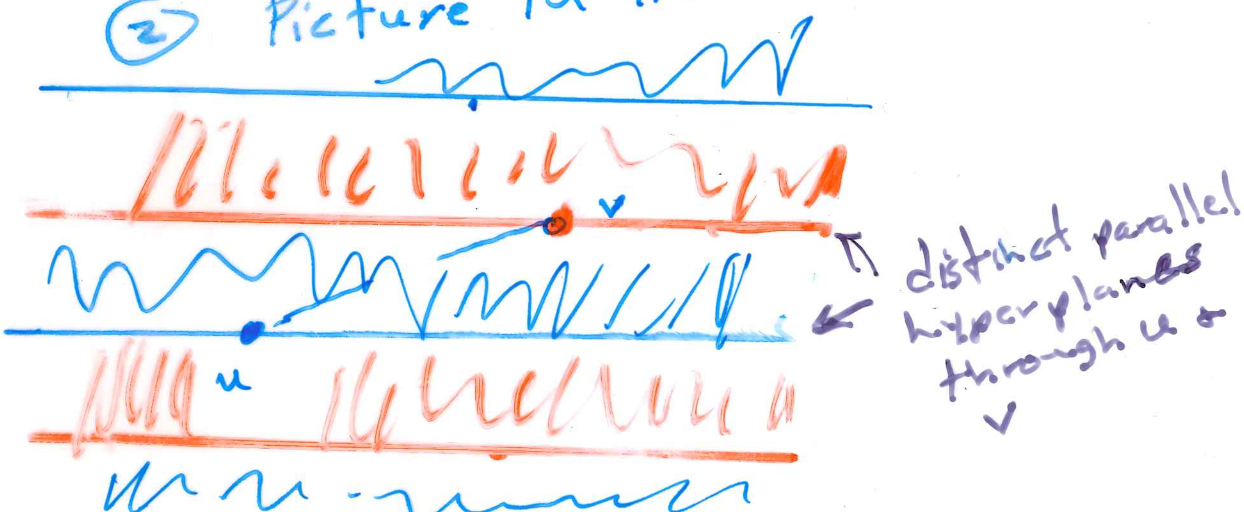
To see that this problem is almost, but not quite, trivial, let's back up to a truly trivial problem:

Given $D \subseteq \mathbb{R}^n$, $|D| = 2$, can $\mathbb{R}^n$ be colored with 2 colors so that no set $q + D$, $q \in \mathbb{R}^n$, is monochromatic?

$n = 1$ ✓

$n > 1$: ① Given $D = \{u, v\}$, can color each line $L = \{w + t(v-u) \mid t \in \mathbb{R}\}$ with 2 colors so that the distance $\underbrace{\cancel{d}}_{|v-u|}$ is forbidden on the line.

② Picture in $\mathbb{R}^2$



Tile $\mathbb{R}^n$ with half-open slabs, color alternately with blue and orange.

[In fact, all the 2-colorings under ② can be seen as special cases of colorings under ①.]

~~Corollary~~ Theorem. Given $F_1, F_2 \subseteq \mathbb{R}^n$, $n > 1$,

if $u_1, v_1 \in F_1$, $u_2, v_2 \in F_2$ and

$w_1 = v_1 - u_1$, $w_2 = v_2 - u_2$ are linearly independent, then there is a 2-coloring of $\mathbb{R}^n$ s.t. no translate of either F_1 or F_2 is monochromatic.

Proof. Let $F_2' = u_1 - u_2 + F_2$

Then $u_2' = u_1 - u_2 + u_2 = u_1$

$v_2' = u_1 - u_2 + v_2$

$\Rightarrow v_2' - u_2' = v_2 - u_2 = w_2$

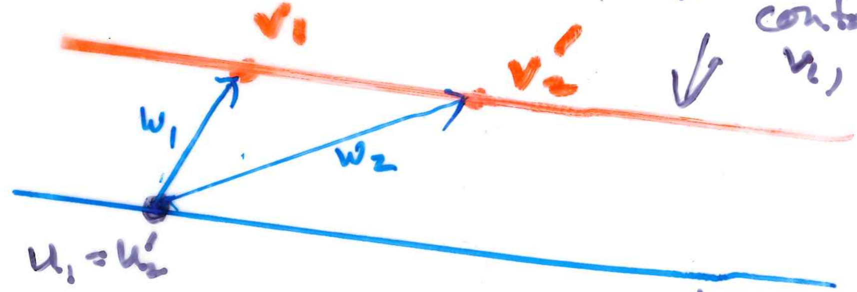
$\quad\quad \parallel$

$v_2' - u_1$

Suffices to find a 2-coloring of $\mathbb{R}^n$ which forbids monochromatic translates of $\{u_1, v_1\} \subseteq F_1$ and of $\{u_2', v_2'\} \subseteq F_2'$.

Picture in $\mathbb{R}^2$:

hyperplane H_1 containing v_1, v_2', but not u_1



$H_2 = H_1 - w_1$

Color $\mathbb{R}^n$ with the "alternating slab" 2-coloring determined by H_1 and H_2.

Remark: For $n > 2$ can get an analogous theorem about coloring $\mathbb{R}^n$ with 2 colors so as to forbid monochromatic translates of $F_1, \ldots, F_k$, $2 < k \le n$.

Corollary Suppose $n \ge 2$, $F_1, F_2 \subseteq \mathbb{R}^n$, and $|F_i| \ge 2$, $i = 1, 2$. There is a 2-coloring of $\mathbb{R}^n$ forbidding monochromatic translates of each F_i unless, for some 1-dimensional subspace L of $\mathbb{R}^n$, each F_i lies on a translate of L.

Given $F_1, F_2 \subseteq \mathbb{R}^n$, each lying on a translate of L. Then each translate of each F_i lies on a translate of L, and each translate of L contains translates of F_1 and F_2. WOLOG, suppose $F_1, F_2 \subseteq L$. Then $\mathbb{R}^n$ is 2-colorable so that translates of F_1 and of F_2 are not monochromatic if and only if $L \simeq \mathbb{R}$ is 2-colorable so that monochromatic translates of F_1 and F_2 in L are forbidden.

So -- the question of whether or not $\mathbb{R}^n$ can be 2-coloved so that monochromatic translates of two specified sets are forbidden reduces to the same question for $n = 1$.

Forbidding monochromatic <u>translates</u> of $D = \{a, b\} \subseteq \mathbb{R}$, $a \neq b$, is the same as forbidding the <u>distance</u> $|b-a|$.

If $F \subseteq \mathbb{R}$ and monochromatic translates of a subset $S \subseteq F$ are forbidden by a coloring of $\mathbb{R}$, then monochromatic translates of F are forbidden by the coloring.

Therefore, if $F_1, F_2 \subseteq \mathbb{R}$, $a_1, b_1 \in F_1$, $a_2, b_2 \in F_2$, and the distances $d_1 = |b_1 - a_1|$, $d_2 = |b_2 - a_2|$ can be forbidden by a 2-coloring of $\mathbb{R}$, then that 2-coloring forbids monochromatic translates of each F_i.

<u>Lemma</u> Suppose that $d_1, d_2 > 0$. There is a 2-coloring of $\mathbb{R}$ forbidding both distances d_1, d_2 $\Longleftrightarrow$ $\frac{d_1}{d_2}$ is either irrational, or rational of type $\frac{\text{odd}}{\text{odd}}$.

This further reduces the question to deciding for which sets $F_1, F_2 \subseteq \mathbb{Z}$,

$\min(|F_1|, |F_2|) \geq 2$

$\max(|F_1|, |F_2|) \geq 3$, is there a 2-coloring of $\mathbb{Z}$ which forbids monochromatic translates of each F_i?

PJ's rash conjecture (2001?)

$T_1, T_2 \subseteq \mathbb{Z}$, $|T_1| = |T_2| = 3 \Rightarrow$ there is a 2-coloring of $\mathbb{Z}$ which forbids monochromatic translates of each T_i.

Counterexample due to Balász Gosztonyi (2010): $T_1 = \{0, 2, 6\}$, $T_2 = \{0, 1, 8\}$

Relevant: Kemnitz + Marangio (2007) and L. Anderson and PJ (2013): For any $d_1, \ldots, d_k > 0$, $\mathbb{R}$ can be colored with $k+1$ colors so that each distance d_i is forbidden.

Corollary: For any $\underbrace{F_1, \ldots, F_k \subseteq \mathbb{R}}_{|F_i| \geq 2,\ i = 1, \ldots, n}$, $\mathbb{R}$ can be colored with $k+1$ colors so that no translate of any F_i is monochromatic.

So for any $T_1, T_2 \subseteq \mathbb{Z}$, $|T_i| \geq 2$, $i = 1, 2$,

$$\chi_t(\mathbb{Z}; T_1, T_2) \in \{2, 3\}$$

$\uparrow$ translate

Now, what?

① Could try to obtain a succinct characterization of those pairs $T_1, T_2 \subseteq \mathbb{Z}$, $|T_1| = |T_2| = 3$, such that $\chi_t(\mathbb{Z}; T_1, T_2) = 3$.

Easier preliminary round: characterize pairs $D = \{0, a\}$, $T = \{0, b, c\}$, $a > 0$, $0 < b < c$, such that $\chi_t(\mathbb{Z}; D, T) = 3$.

Some results obtained by L. Anderson and P.J., and later by K. Binder:

(a) There is an algorithm for checking whether $\chi_t(\mathbb{Z}; D, T) = 2$ or 3. At worst, runs in $a \cdot 2^{a-1}$ steps.

(b) Given b and c, for all a suff. large, $\chi_t(\mathbb{Z}; D, T) = 2$

[If $\underbrace{2 < b \leq c/2 < \frac{a}{2}}_{\text{Can reduce to this; } b \text{ and } c \text{ here may not be the original } b \text{ and } c}$ and $a \geq (c-b)^2$, then $\chi_t = 2$]

② Does there exist $k > 3$ s.t. for all $F_1, F_2 \subseteq \mathbb{Z}$ with $|F_i| \geq k$, $i = 1, 2$, $\chi_t(\mathbb{Z}; F_1, F_2) = 2$?