Additive bases in groups

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Part I: Additive bases in $\mathbf{N}$

Additive bases in **N**

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$$hA = \{a_1 + \cdots + a_h : a_1, \ldots, a_h \in A\}.$$
- For two sets X, Y, we write $X \sim Y$ if their symmetric difference $(X \setminus Y) \cup (Y \setminus X)$ is finite.

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$$\mathrm{ord}^*(A) = h.$$

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- If $A = \{n^3 : n \geq 0\}$, then $4 \leq \mathrm{ord}^*(A) \leq 7$.
- If $A = \{n^k : n \geq 0\}$, then $\mathrm{ord}^*(A) = G(k) \leq (k + o(1)) \log k$.
- If A is the set of primes, then $\mathrm{ord}^*(A) \leq 4$ (conjectured to be 3).

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Question

What are some properties that a generic basis of $\mathbf{N}$ must satisfy?

Removing one element

Question (Erdős-Graham 1980)

Let A be a basis of $\mathbf{N}$ and $a \in A$. When is $A \setminus \{a\}$ still a basis (of some possibly different order)?

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Clearly, we can't expect all the elements to be regular. For example, if

$$A = \{1\} \cup \{3, 6, 9, \ldots\}$$

then 1 is exceptional and all the other elements are regular.

The number of exceptional elements

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Theorem (Grekos 1987)

If $hA \sim \mathbf{N}$, then the number of exceptional elements of A is $\leq h - 1$.

The number of exceptional elements

Thus we can define the function

$$E(h) = \max_{hA \sim \mathbf{N}} \#\,\text{exceptional elements of } A.$$

and study the behavior of $E(h)$ when $h \to \infty$.

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Theorem (Plagne 2008)

As $h \to \infty$, we have

$$E(h) \sim 2\sqrt{\frac{h}{\log h}}.$$

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Clearly, $X(h) \geq h$.

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Clearly, $X(h) \geq h$. Erdős-Graham also showed that

$$X(h) \geq (1/4 + o(1))h^2.$$

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We know $X(1) = 1, X(2) = 4$, $X(3) = 7$, but $X(4)$ is unknown.

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Clearly $S(h) \leq X(h)$. Grekos conjectured that $S(h) < X(h)$ for any $h \geq 2$.

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Question

Is there a constant β such that $S(h) \sim \beta h$ as $h \to \infty$?

Part II: From $\mathbf{N}$ to groups

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- It makes sense to study the analogs of the functions E, X, S when **N** is replaced by G. A priori, we don't even know if these functions are well defined!
- From now on, we assume that G is an infinite abelian *group*.
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- Joint work with Victor Lambert and Alain Plagne.

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2. *$hA = G$.*

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and we have

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as $n \to \infty$.

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and this is best possible, since for $G = \mathbf{F}_p[t]$, *we have*

$$E_G(h) = \left\lfloor \frac{h-1}{p-1} \right\rfloor.$$

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Recall that in $\mathbf{N}$, the function $X(h)$ is defined by

$$X(h) = \max_{hA \sim \mathbf{N}} \max\{\mathrm{ord}^*(A \setminus \{a\}) : a \in A \text{ is regular}\}$$

and

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We don't know if this function is well defined! For fixed G, we don't know if $X_G(h)$ is finite, not to mention if this can be bounded in terms of h alone.

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Theorem

Suppose the quotient $G/m \cdot G$ is finite for any $1 \leq m \leq h$. Then

$$X_G(h) \leq h^2 + h \cdot \max_{1 \leq m \leq h} \Omega\left(|G/m \cdot G|\right) + h - 1.$$

Here $m \cdot G = \{mx : x \in G\}$ and $\Omega\left(p_1^{\alpha_1} \cdots p_k^{\alpha_k}\right) = \alpha_1 + \cdots + \alpha_k$.

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Groups G such that $G/m \cdot G$ is finite for any m include:

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- Finitely generated abelian groups.
- $\mathbf{Z}_p$.

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We also have another quadratic lower bound for $X_G(h)$ for another class of groups.

Question

Is $X_G(h)$ quadratic in h?

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2. *For any h,*
$$X_G(h) \geq 2h - 3p + 3.$$

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1. *For any $h \geq p$,*
$$X_G(h) \leq ph + p - 1.$$
2. *For any h,*
$$X_G(h) \geq 2h - 3p + 3.$$
3. *In particular, if $p = 2$, then $X_G(h) \sim 2h$ as $h \to \infty$.*

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1. $3 \leq X_G(2) \leq 5$.

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For any G, we have

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Recall that in $\mathbf{N}$ we have $X(2) = 4$ and $X(3) = 7$.

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A priori, we don't know if $S_G(h)$ is finite. (In **N**, S is finite since $S(h) \leq X(h)$ and $X(h)$ is known to be finite.)

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Theorem

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