# The Beta Invariant and Chromatic Uniqueness of Wheels

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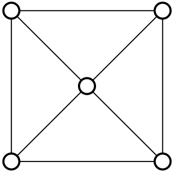
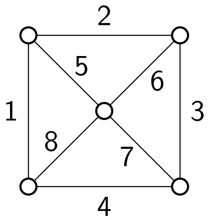
# What is a Matroid?

Matroid $M$ is an ordered pair $(E, \mathcal{I})$ consisting of a finite set $E$ and a collection $\mathcal{I}$ of subsets of $E$ having the three properties.
(I1) $\emptyset \in \mathcal{I}$
(I2) $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$.
(I3) If $I_1$ and $I_2$ are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e$ of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$

# Graph and Matroid



Graph $G$ Cycle Matroid of $G$, $M(G)$




Graph $G$

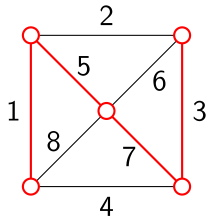
Matroid $M(G)$

Edges

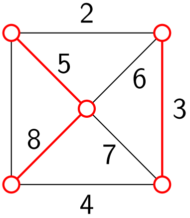
Elements

$e(G) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$E(M(G)) = \{1, 2, 3, 4, 5, 6, 7, 8\}$



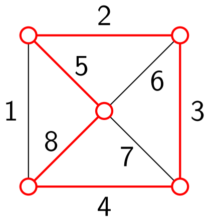

| | |
|---|---|
| Graph $G$ | Matroid $M(G)$ |
| Tree, Forest | Independent set |
| $H$ is a tree | $\{1, 3, 5, 7\}$ is independent |




Graph $G$ | Matroid $M(G)$

Tree, Forest | Independent set

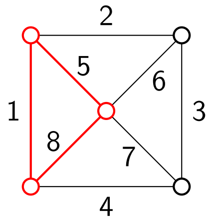
$H$ is a forest | $\{3, 5, 8\}$ is independent set




Graph $G$ | Matroid $M(G)$

A Cycle | Circuit

$H$ is a cycle | $\{2,3,4,5,8\}$ is a circuit




Graph $G$

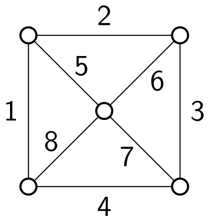
Matroid $M(G)$

A Cycle

Circuit

$H$ is a cycle

$\{1, 5, 8\}$ is a circuit




Graph $G$

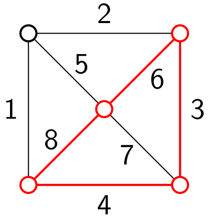
Matroid $M(G)$

$|v(G)| - 1$

Rank

$5 - 1 = 4$

$r(M) = 4$




Graph $G$ Matroid $M(G)$

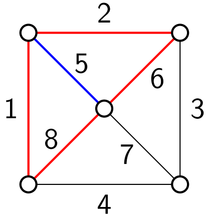
Let $e(H) = X = \{3, 4, 6, 8\}$

$V(H) - 1$ Rank of the set $X$

$4 - 1 = 3$ $r_M(X) = 3$

Closure of $X$ in $M$,

$$cl_M(X) = \{x \in E(M) : r(X \cup x) = r(X)\}$$



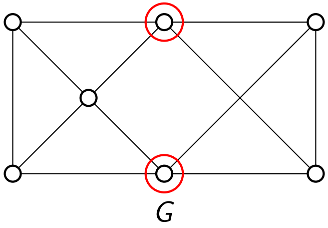

Let $X = \{1, 2, 6, 8\}$.

Then $cl_{M(G)}(X) = \{1, 2, 6, 8, 5\}$

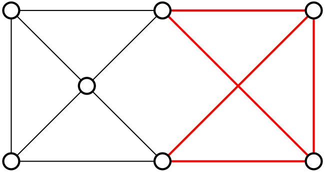
A set $X \subseteq E(M)$ is a $k$-separation if

$r(X) + r(E(M) - X) - r(M) \leq k - 1$ and $|X|, |E(M) - X| \geq k$.

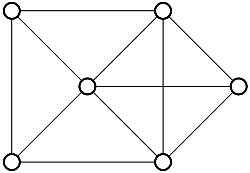
$M$ is $n$-connected if for all $k$ in $\{1, 2, \cdots, n-1\}$, it has no $k$-separation.




Vertex cut of size 2




3-separation of $M(G)$



$H$

$H$ is a 3-connected graph.

$M(H)$ is a 3-connected matroid.

# Beta Invariant

Beta Invariant, Crapo (1967)

The numerical value $\beta$, defined for each finite matroid $M$ with rank function $r$ is defined by

$$\beta(M) = (-1)^{r(M)} \sum_{X \subseteq E(M)} (-1)^{|X|} r(X)$$

# Characteristic Polynomial

Characteristic Polynomial of a Matroid $M$,

$$p(M; z) = \sum_{x \subseteq E(M)} (-1)^{|X|} z^{r(M) - r(X)}.$$

Let $M(G)$ be a cycle matroid derived from a graph $G$ and $P(M(G); z)$ be its characteristic polynomial. If $G$ has $\omega(G)$ connected components, then the chromatic polynomial of $G$ is

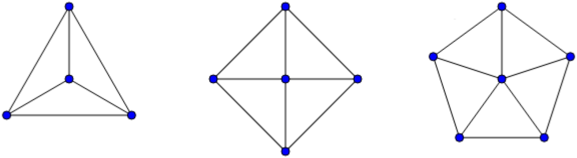
$$P_G(z) = z^{\omega(G)} p(M(G); z).$$

Let $M(G)$ be a 3-connected cycle matroid. Then,

$$\beta(M(G)) = (-1)^{r(M(G))+1} \frac{d}{dz} P(M(G); z) \bigg|_{z=1}$$

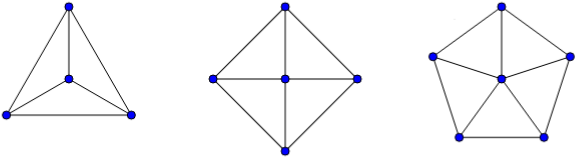
$$= (-1)^{r(M(G))+1} \frac{d}{dz} \frac{P_G(z)}{z} \bigg|_{z=1}$$

# Wheel (Graph)



$W_4$, $W_5$ and $W_6$

# Wheel (Matroid)

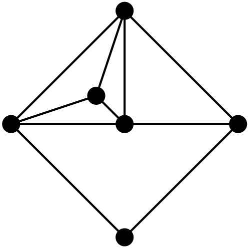


$W_3$, $W_4$ and $W_5$

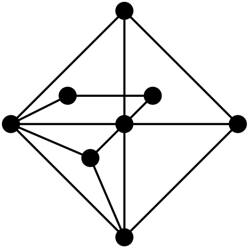
# Chromatic Uniqueness of Wheels

**Theorem**

*(1) (Chao, Whitehead, 1979)* $W_5$ *is not chromatically unique.*
*(2) (Xu, Li, 1984) For every even* $n \geq 4$*, the wheel graph* $W_n$ *is chromatically unique.*
*(3) (Xu, Li, 1984)* $W_7$ *is not chromatically unique.*
*(4) (Read, 1988), (Xu, Li, 1992)* $W_9$ *is chromatically unique.*
*(5) (Al-Rekaby, Khalaf, 2014), (Azarija, 2016)* $W_{11}$ *and* $W_{13}$ *are chromatically unique.*



$G$



$H$

$$P_G(\lambda) = P_{W_5}(\lambda) \text{ and } P_H(\lambda) = P_{W_7}(\lambda)$$

## Theorem (Lee, Wu)

*Let $\mathcal{M}_n$ be the set of 3-connected matroids with $\beta = n$, $n > 1$. Then $|E(M)| = max\{|E(N)| : N \in \mathcal{M}_n\}$ if and only if $M \cong W_{n+1}$.*

## Corollary (Lee, Wu)

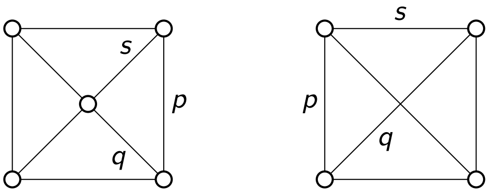
*Let $M$ be a 3-connected matroid such that $|E(M)| = |E(W_n)|$ and $\beta(M) = \beta(W_n)$. Then $M \cong W_n$.*

## Proposition

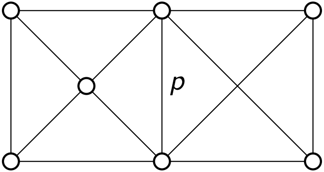
*Let $M$ and $N$ be simple matroid such that $P(M;\lambda) = P(N;\lambda)$. Then $r(M) = r(N)$ and $|E(M)| = |E(N)|$.*

## Corollary (Lee, Wu)

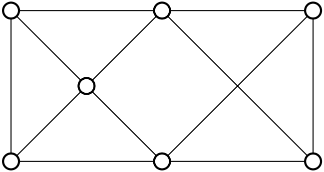
*Let $G$ be a 3-connected graph such that $P_G(\lambda) = P_{W_n}(\lambda)$. Then $G \cong W_n$.*




$G$ $H$



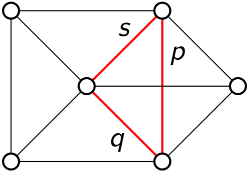

$P(M(G), M(H))$



$M(G) \oplus_2 M(H)$

$P(M(G), M(H))$
Parallel connection of $M(G)$ and $M(H)$ with respect to the base point $p$.

$M(G) \oplus_2 M(H)$
2-sum of matroids $M(G)$ and $M(H)$.




$$P_{\{p,s,q\}}(M(G), M(H))$$

$$P_{\{p,s,q\}}(M(G), M(H))$$

Parallel connection of $M(G)$ and $M(H)$ across a triangle $\{p, s, q\}$.

Brylawski(1971) showed that,

$$\beta(M_1 \oplus_2 M_2) = \beta(M_1)\beta(M_2)$$

.

Using Brylawski's result with the deletion/contraction formula for the beta invariant, we can obtain,

$$\beta(P(M_1, M_2)) = \beta(M_1)\beta(M_2)$$

.

## Theorem (Lee, Wu)

*Let* $M = P_T(M_1, M_2)$ *where* $T$ *is a triangle. Then,*

$$\beta(P_T(M_1, M_2)) = \beta(M_1)\beta(M_2)$$

.

# Idea of proof

Let $\overline{Cl}(M,p) = \{A \subseteq E(M)\backslash p \text{ such that } p \notin cl_M(A)\}$.

$X \in \overline{Cl}(M,p)$ such that $X \cup s$ and $X \cup q \in \overline{Cl}(M,p)$

$Y \in \overline{Cl}(M,p)$ such that $Y \cup s$ and $Y \cup q \notin \overline{Cl}(M,p)$

$A \in \overline{Cl}(M_1,p)$ such that $A \cup s$ and $A \cup q \in \overline{Cl}(M_1,p)$

$B \in \overline{Cl}(M_2,p)$ such that $B \cup s$ and $B \cup q \in \overline{Cl}(M_2,p)$

$C \in \overline{Cl}(M_2,p)$ such that $C \cup s$ and $C \cup q \notin \overline{Cl}(M_2,p)$

Let $M = P_T(M_1, M_2)$ where $T = \{p, s, q\}$.

$$\beta(M) = (-1)^{r(M)} \sum_{A \subseteq E(M)} (-1)^{|A|} r_M(A)$$

$$= (-1)^{r(M)} \sum_{X, Y \subseteq E(M)} (-1)^{|X|} + (-1)^{|Y|+1}$$

$$
\begin{aligned}
\beta(M) &= (-1)^{r}\sum\left[(-1)^{|X|}+(-1)^{|Y|+1}\right]\\
&= (-1)^{r_1+r_2-2}\sum\left[(-1)^{|A|+|B|}+(-1)^{|A|+|C|+1}\right]\\
&= (-1)^{r_1+r_2}\sum(-1)^{|A|}\left[(-1)^{|B|}+(-1)^{|C|+1}\right]\\
&= \left[(-1)^{r_1}\sum(-1)^{|A|}\right]\left[(-1)^{r_2}\sum(-1)^{|B|}+(-1)^{|C|+1}\right]\\
&= \beta(M_1)\beta(M_2)
\end{aligned}
$$

# Why 3-connected?

Let $N_1 = P(M_1, M_2) \quad N_2 = M_1 \oplus_2 M_2 \quad N_3 = P_T(M_1, M_2)$

$N_1 \ncong N_2 \ncong N_3$

$N_1$, $N_2$ connected but not 3-connected. $N_3$ is 3-connected.

$\beta(N_1) = \beta(N_2) = \beta(N_3) = \beta(M_1)\beta(M_2)$

Let $M$ be a matroid and $t(M)$ be tutte polynomial of $M$. Then $\beta(M)$ is the coefficient of $x$ or $y$. Let,

$$A_1 = t(M_1/q) + t(M_1/p) + t(M_1/s).$$

$$A_2 = t(M_2/q) + t(M_2/p) + t(M_2/s).$$

$$A_3 = t(M_1/q)t(M_2/q) + t(M_1/p)t(M_2/p) + t(M_1/s)t(M_2/s)$$

Andrzejak (1997), (Oxley's formula)

$$
\begin{aligned}
t(P_T(M_1, M_2)) = \\
&= (xy - x - y)^{-1}(xy - x - y - 1)^{-1} \\
&\quad [(xy - x - y - 1)yA_3 \\
&\quad +2y^3[t(M_1)t(M_2/q/p/s) + t(M_2)t(M_1/q/p/s)] \\
&\quad +y^2 A_1 A_2 \\
&\quad +y(1-y)[t(M_1)A_2 + t(M_2)A_1] \\
&\quad -y^3(1+x)[t(M_1/q/p/s)A_2 + t(M_2/q/p/s)A_1] \\
&\quad +y^3(x^2 + x + y + 3xy)t(M_1/q/p/s)t(M_2/q/p/s) \\
&\quad +(y-1)^2 t(M_1)t(M_2)].
\end{aligned}
$$

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