On the maximum degree of path-pairable graphs

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5th Annual Mississippi Discrete Math Workshop
Oxford, MS

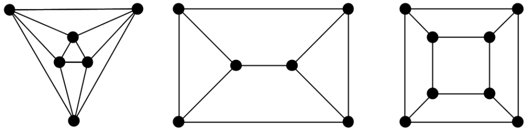
Path-Pairable Graphs

Definition

For k fixed, an undirected graph G is k -path-pairable if, every ordered set of $2k$ pairwise disjoint vertices $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$ there exist edge-disjoint paths $P_1, \ldots, P_k$ such that each P_i is an $s_i t_i$-path.

Definition

A graph G on $2k$ vertices is *path-pairable* if it is k-path-pairable.



Asymptotic Behavior, Scalability

Observations

- Unlike many related parameters (linkedness, weak-linkedness), path-pairability does not require high edge density.
- On the other hand, path-pairable graphs all seem to have reasonable high maximum degree.

The Maximum Degree Problem

Theorem (Faudree, Gyárfás,Lehel, 1991)

There exist k-path-pairable graphs with maximum degree $\Delta = 3$ for arbitrary high values of k.

Theorem (Faudree, Gyárfás,Lehel, 1992)

If G is a path-pairable graph on n vertices with maximum degree Δ, then $n \leq 2\Delta^{\Delta}$.

Proof

- Let $d = \log_{\Delta} \frac{n}{2}$.
- We can choose a pairing of the vertices of distance d or more (why?).
- We need $\frac{n}{2} \cdot \log_{\Delta} \frac{n}{2}$ edges to built the paths but only have $\frac{n}{2} \cdot \Delta$.

The Maximum Degree Problem - Constructions

$$\frac{\log n}{\log \log n} < \Delta_{min} < \dots$$

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Advances

- Csaba, Faudree, Gyárfás, Lehel, Schelp (1991): $\Delta_{min} \leq 3 \cdot \sqrt{n}$
- Lehel, Kubicza, Kubicky (1999): $\Delta_{min} \leq 2 \cdot \sqrt{n}$.
- M. (2013): $\Delta_{min} \leq 2\sqrt{2} \cdot \sqrt{n}$.
- M. (2014): $\Delta_{min} \leq \sqrt{n}$.
- Győri, Mezei, M. (2016): $\Delta_{min} \leq 5.2 \cdot \log n$

The Maximum Degree Problem - Constructions

Conjecture(Csaba, Faudree, Gyárfás, Lehel, Schelp, 1991)

Hypercubes of odd dimension are path-pairable.

Conjecture(Lehel, Kubicza, Kubicky, 1999)

Sufficiently large three dimensional complete grid graphs are path-pairable.

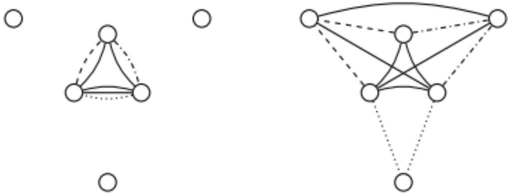
Terminal-Pairability in Graphs

Definition

Given a simple undirected graph G and an undirected multigraph D with $V(D) = V(G)$ we say that G can realize the edges $e_1, \ldots, e_{|E(D)|}$ of D if there exist edge disjoint paths $P_1, ..., P_{|E(D)|}$ in G joining the endpoints of $e_1, \ldots, e_{|E(D)|}$, respectively.

Definition

A graph G is *terminal-pairable* with respect to a family $\mathcal{F}$ of demand graphs on $V(G)$ if every demand graph in $\mathcal{F}$ can be realized by G.



Terminal-Pairability in Graphs

Problem (Csaba, Faudree, Gyárfás, Lehel, Schelp, 1991))

Let $G = K_n$ and let $\mathcal{F}_t = \{D : \Delta(D) <= t\}$. What is the maximum of t for which G is terminal-pairable w.r.t. $\mathcal{F}_t$?

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K_n is terminal-pairable w.r.t. $\mathcal{F}_t$ for $t \leq \frac{n}{3} - c$.

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Theorem (Girão, M., 2016)

K_n is not terminal-pairable w.r.t. $\mathcal{F}_t$ for $t > \frac{13}{27}n + c$.

Back to the PP Δ Problem

Theorem (Győri, Mezei, M.)

Let $G = K_t^n$ and let $D = (V(D), E(D)$ be a demand graph with $V(D) = V(K_t^n)$ and $\Delta(D) \leq \lfloor \frac{t}{6} \rfloor - 2$ even. Then every demand edge of D can be assigned a path in G joining the same endpoints such that the system of paths is edge-disjoint.

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Corollary

If $t \geq 24$, K_t^n is path-pairable.

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$\Delta_{min} \leq 5.2 \cdot \log n$.

Open Problems

- $\frac{\log n}{\log \log n} < \Delta < c \cdot \log n$
- Q_n is path-pairable.
- Sharp bound on the terminal-pairability of complete graphs.

- Path-pairable planar graphs.
- Δ-forcing in k-path-pairable graphs.

THANK YOU FOR YOUR ATTENTION!