Oriented hypergraphic matrix-tree and Sachs type theorems

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Discrete Math Workshop

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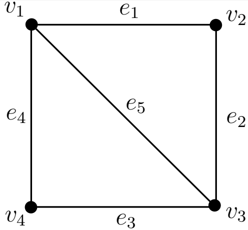
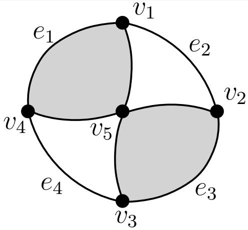
Definitions

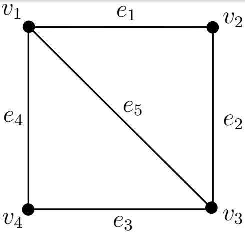
- A *graph* is a collections of labeled 2-subsets of V.
- A *set system* is a collections of labeled subsets of $\mathcal{P}(V)$.

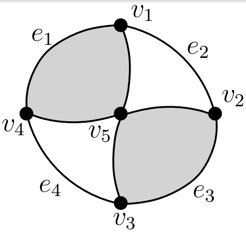




Definitions

- A *directed graph* consists of disjoint sets V and E, and a pair of functions (σ, τ) from $E \to V$.
- An *incidence hypergraph* consists of disjoint sets V, E, and I and a function $\iota : I \to V \times E$.





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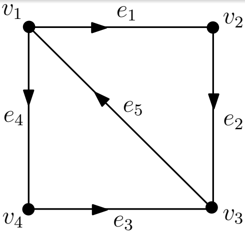

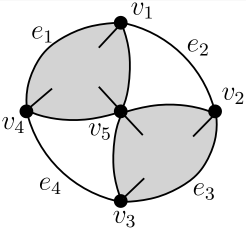

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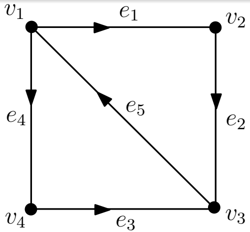
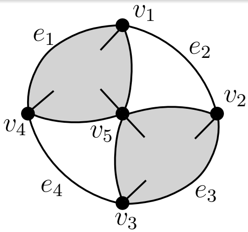
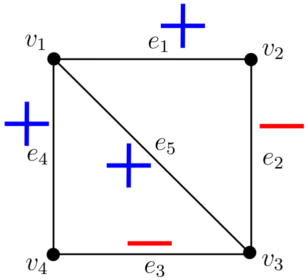
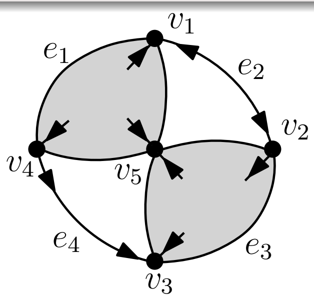




Definitions

- A *signed graph* is a graph with an edge signing function $\psi : E \to \{+1, -1\}$.
- An *oriented hypergraph* is an incidence hypergraph is an incidence signing function $\sigma : I \to \{+1, -1\}$.





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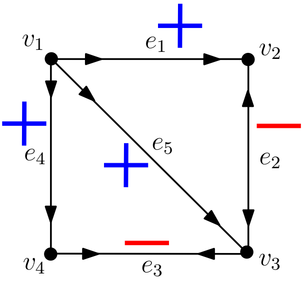
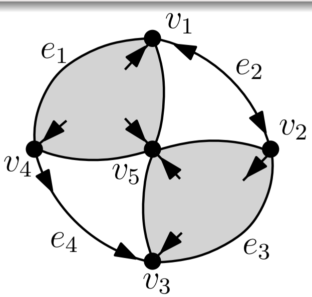
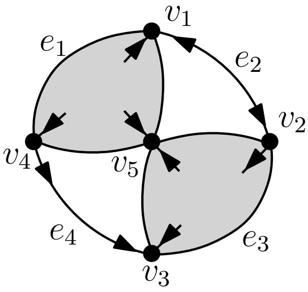




Associated Matrices

- **Incidence Matrix:** H_G
- Degree Matrix: D_G
- Adjacency Matrix: A_G
- Laplacian Matrix: $L_G := D_G - A_G = \mathrm{H}_G \mathrm{H}_G^T$



$$\mathrm{H}_G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- Incidence Matrix: H_G
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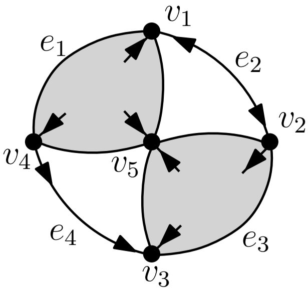

$$D_G = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

- Incidence Matrix: H_G
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- **Adjacency Matrix:** A_G
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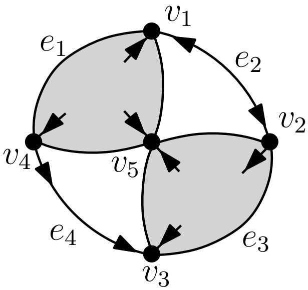
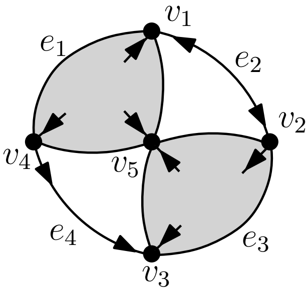


$$A_G = \begin{bmatrix} 0 & -1 & 0 & -1 & -1 \\ -1 & 0 & +1 & 0 & +1 \\ 0 & +1 & 0 & +1 & -1 \\ -1 & 0 & +1 & 0 & -1 \\ -1 & +1 & -1 & -1 & 0 \end{bmatrix}$$

- Incidence Matrix: H_G
- Degree Matrix: D_G
- Adjacency Matrix: A_G
- **Laplacian Matrix:** $L_G := D_G - A_G = \mathrm{H}_G \mathrm{H}_G^T$



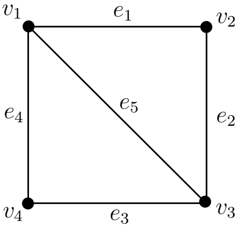
$$L_G = \begin{bmatrix} 2 & 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 1 \\ 1 & 0 & -1 & 2 & 1 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix}$$

Graph Theorems

Theorem (Sachs' Theorem)

Let G be a graph, $\mathscr{B}_i$ be the set of basic figures with exactly i isolated vertices, $tf(B)$ be the total number of elementary figures, and $cf(B)$ be the number of circuits in B.

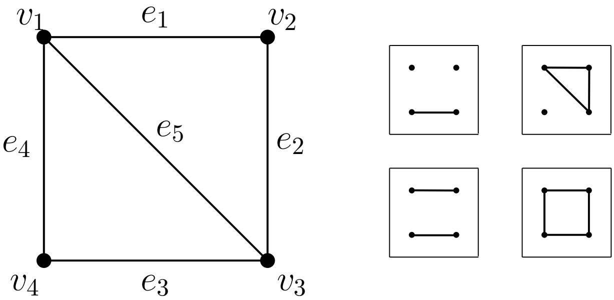
$$\chi(A_G, x) = \sum_{i=0}^{|V|} \Big(\sum_{B \in \mathscr{B}_i} (-1)^{tf(B)} (2)^{cf(B)} \Big) x^i.$$



$$\chi(A_G, x) = x^4 - 5x^2 - 4x$$

Definitions

- An *elementary figure* is a circuit or a path of length 1.
- A *basic figure* is a disjoint union of elementary figures.

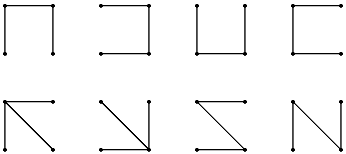


Theorem (Matrix-tree Theorem)

Let G be a graph, $T(G)$ be the number of spanning trees of G, and L_{ij} be the ij-minor of the Laplacian, then

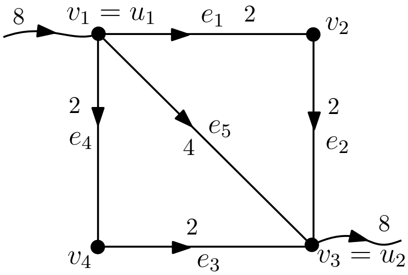
$$det(L_{ij}) = (-1)^{i+j} T(G).$$

$$L_G = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$



Theorem (Tutte's Transpedance Theorem)

The ordered second cofactors produce an edge labelling that satisfies Kirchhoff's Laws. Moreover, the initial energy is the in equal to the first cofactor.

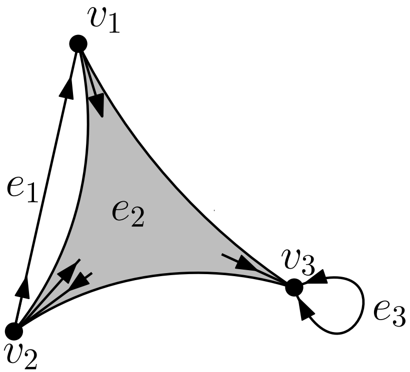


$$L_G = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Incidence Intricacies

Definitions

- A *k-weak walk* is an incidence preserving embedding of $\overrightarrow{P}_k$ into G.
- A *backstep* is a non-incidence-monic weak walk of length 1.



Theorem

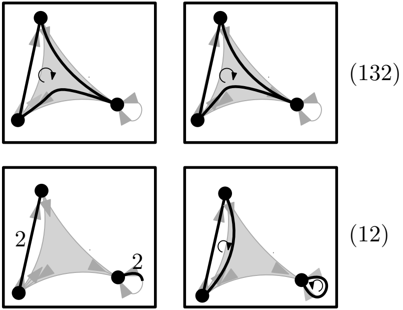
Let G be an oriented hypergraph,

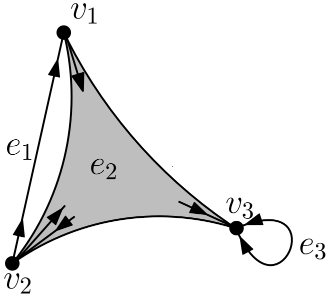
1. H_G *is the half-walk matrix,*
2. $-D_G$ *is the strictly* 1*-weak-walk matrix,*
3. A_G *is the* 1*-~~weak~~-walk matrix,*
4. $-L_G$ *is the* 1*-weak walk matrix,*
5. A_G^k *is the* k*-walk matrix,*
6. $(-1)^k L_G^k$ *is the* k*-weak walk matrix,*

Hypergraphic Sachs-type Theorems

Definitions

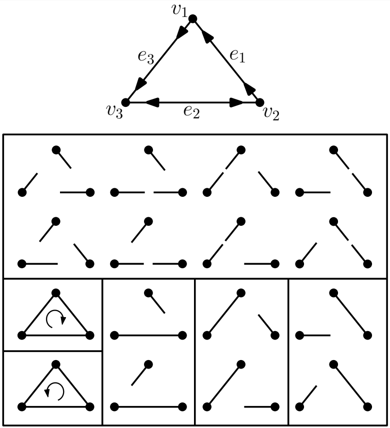
- A *contributor of G* is an incidence preserving map from a disjoint union of $\overrightarrow{P}_1$'s into G defined by $c : \coprod_{v \in V} \overrightarrow{P}_1 \to G$ such that $c(t_v) = v$ and $\{c(h_v) \mid v \in V\} = V$.
- Let $\mathfrak{C}(G)$ denote the set of contributors.

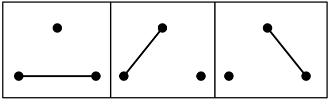




Definition

$\mathfrak{C}_{=k}(G)$ is the set of contributors of G with exactly k backsteps.
$\widehat{\mathfrak{C}}_{=k}(G)$ removes k backsteps.

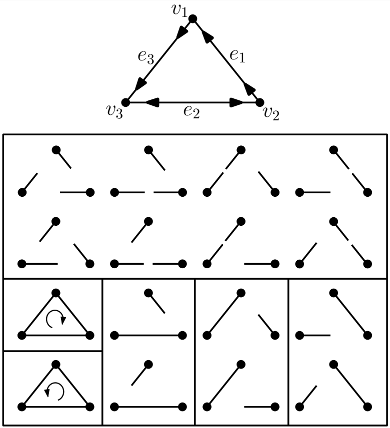


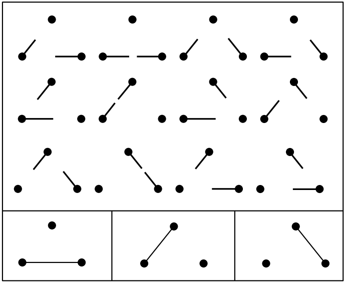


$\widehat{\mathfrak{C}}_{=1}(G)$

Definition

$\mathfrak{C}_{\geq k}(G)$ is the set of contributors of G with at least k backsteps.
$\widehat{\mathfrak{C}}_{\geq k}(G)$ removes k backsteps.





$\widehat{\mathfrak{C}}_{\geq 1}(G)$

Theorem (Chen, Liu, Robinson, R., Wang, 2017)

Let G be an oriented hypergraph with adjacency matrix A_G and Laplacian matrix L_G, then

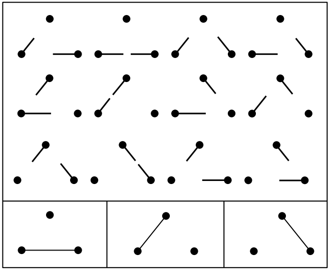
1. $perm(L_G) = \sum_{c \in \mathfrak{C}_{\geq 0}(G)} (-1)^{oc(c)+nc(c)}$,
2. $det(L_G) = \sum_{c \in \mathfrak{C}_{\geq 0}(G)} (-1)^{pc(c)}$,
3. $perm(A_G) = \sum_{c \in \mathfrak{C}_{=0}(G)} (-1)^{nc(c)}$,
4. $det(A_G) = \sum_{c \in \mathfrak{C}_{=0}(G)} (-1)^{ec(c)+nc(c)}$.

Theorem (Chen, Liu, Robinson, R., Wang, 2017)

Let G be an oriented hypergraph with adjacency matrix A_G and Laplacian matrix L_G, then

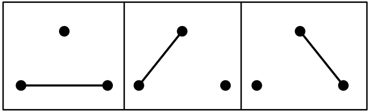
1. $\chi^P(A_G, x) = \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k}(G)} (-1)^{oc(c)+nc(c)} \right) x^k,$
2. $\chi^D(A_G, x) = \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{=k}(G)} (-1)^{pc(c)} \right) x^k,$
3. $\chi^P(L_G, x) = \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{\geq k}(G)} (-1)^{nc(c)+bs(c)} \right) x^k,$
4. $\chi^D(L_G, x) = \sum_{k=0}^{|V|} \left(\sum_{c \in \widehat{\mathfrak{C}}_{\geq k}(G)} (-1)^{ec(c)+nc(c)+bs(c)} \right) x^k.$

$\widehat{\mathfrak{C}}_{\geq 1}(G)$



$\chi^D(L_G, x) = x^3 - 6x^2 + 9x - 4$

$\widehat{\mathfrak{C}}_{=1}(G)$



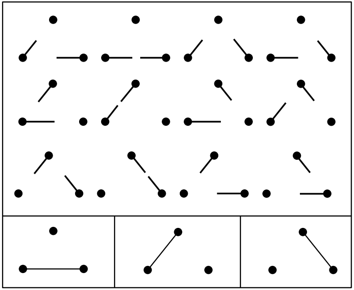
$\chi^D(A_G, x) = x^3 - 3x + 2$

Hypergraphic Matrix-tree-type Theorems

Definitions

- $[M]_{(U;W)}$ is the minor obtained by striking out rows U and columns W from M.
- $\mathfrak{C}(U;W;G)$ is the set of all $c : \coprod_{u \in \overline{U}} \overrightarrow{P}_1 \to G$ with $p(t_u) = u$ and $\{p(h_u) \mid u \in \overline{U}\} = \overline{W}$.

$U = W, \ |U| = 1$



Theorem (Robinson, R., Schmidt, Shroff, 2017)

Let G be an oriented hypergraph with adjacency matrix A_G and Laplacian matrix L_G, then

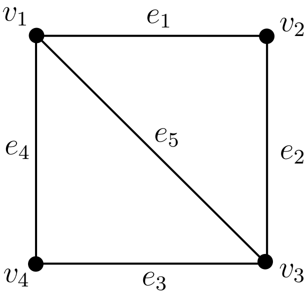
1. $$perm([L_G]_{(U;W)}) = \sum_{c \in \mathfrak{C}(U;W;G)} (-1)^{on(c)+nn(c)},$$
2. $$det([L_G]_{(U;W)}) = \sum_{c \in \mathfrak{C}(U;W;G)} \varepsilon(c) \cdot (-1)^{on(c)+nn(c)},$$
3. $$perm([A_G]_{(U;W)}) = \sum_{c \in \mathfrak{S}(U;W;G)} (-1)^{nn(c)},$$
4. $$det([A_G]_{(U;W)}) = \sum_{c \in \mathfrak{S}(U;W;G)} \varepsilon(c) \cdot (-1)^{en(c)+nn(c)}.$$

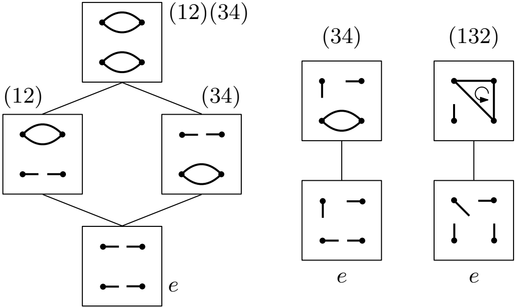
Where $\varepsilon(c)$ is the number of inversions in the natural bijection from $\overline{U}$ to $\overline{W}$.

Applications

Theorem (Robinson, R., Schmidt, Shroff, 2017)

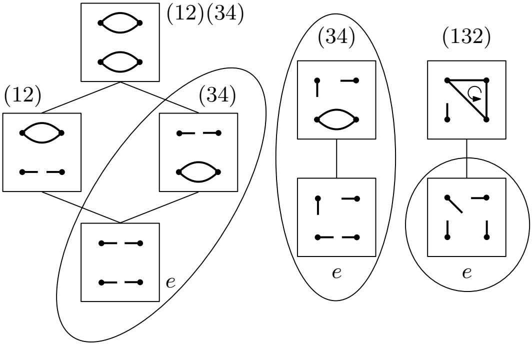
"Activation classes" of a bidirected graph are boolean.





Definitions

- Two contributors c and d are *uw-equivalent* if $c(h_u) = d(h_u) = w$.
- The $(u;w)$-*cut of activation class* $\mathcal{A}$ is the subclass of $\mathcal{A}/\sim_{uw}$ where each element has $c(h_u) = w$.



Lemma (Robinson, R., Schmidt, Shroff, 2017)

If G is a signed graph, then

$$\det(L_G) = \sum_{c \in \mathfrak{M}^-} 2^{nc(c)}.$$

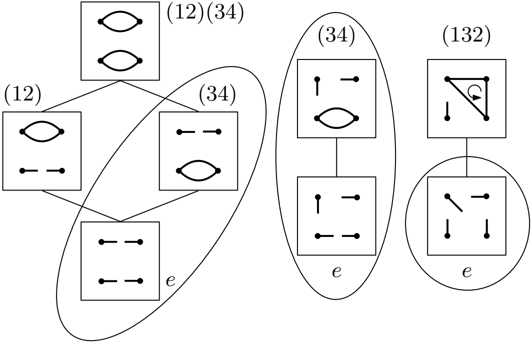
Where $\mathfrak{M}^-$ is the set of maximal elements from the positive-circle-free activation classes.

Corollary

If G is a balanced signed graph, then $\det(L_G) = 0$.

Lemma (Robinson, R., Schmidt, Shroff, 2017)

If G is a bidirected graph, then the set of elements in all single-element $\widehat{\mathcal{A}}_{\neq 0}(u; w; G')$ is activation equivalent to the set of spanning trees of G.



Theorem (Chen, Liu, Robinson, R., Wang, 2017)

Let G be an oriented hypergraph with no isolated vertices or 0-edges with Laplacian matrix L_G, then

1. $-|\mathfrak{C}(G)| < perm(L_G) \leq |\mathfrak{C}(G)|$, *and* $perm(L_G) = |\mathfrak{C}(G)|$ *if, and only if, G is extroverted or introverted,*
2. $-|\mathfrak{C}(G)| < det(L_G) \leq |\mathfrak{C}(G)|$, *and* $det(L_G) = |\mathfrak{C}(G)|$ *if, and only if, the connected components of G consist of bouquets of introverted or extroverted edges.*

Theorem (Chen, Liu, Robinson, R., Wang, 2017)

If G is a balanced signed graph, then $perm(A_G)$ is maximal and equals $|\mathfrak{C}_{=0}(G)|$.

Insight into object comparisons and naturality

Definitions (Graph-like Categories)

1. Quivers: $\mathfrak{Q} := (id_{Set} \downarrow \Delta^{\star}\Delta)$
2. Incidence Structures: $\mathfrak{R} := (id_{Set} \downarrow \Delta^{\star})$
3. Set Systems: $\mathfrak{H} := (id_{Set} \downarrow \mathcal{P})$
4. Multigraphs: $\mathfrak{M}$ is the coreflective subcategory of $\mathfrak{H}$ with set size restricted to 2.

	$\mathfrak{R}$	$\mathfrak{Q}$	$\mathfrak{H}$	$\mathfrak{M}$
Limits	Yes	Yes	Yes	Yes
Colimits	Yes	Yes	Yes	Yes
Subobject classifier	Yes	Yes	Yes	Yes
Cartesian closed	Yes	Yes	No	No
Projective cover	Yes	Yes	No	Yes
Generation family size	3	2	class	2

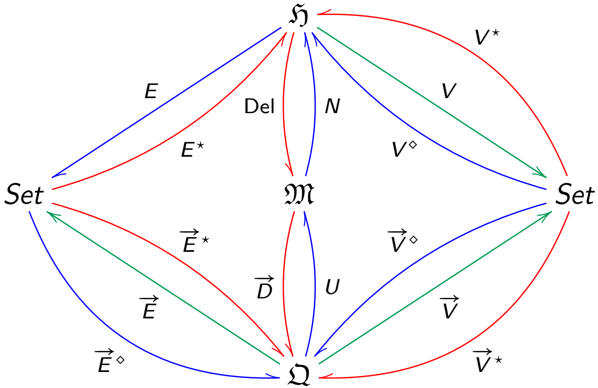
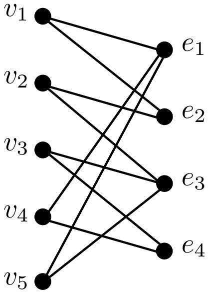
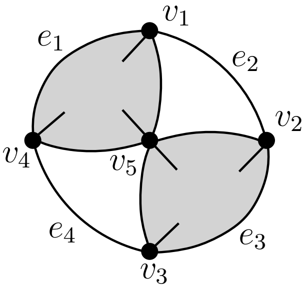


Figure: Full functorial diagram for $\mathfrak{Q}$, $\mathfrak{M}$, and $\mathfrak{H}$



Ω_G



G

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$\Gamma(G)$

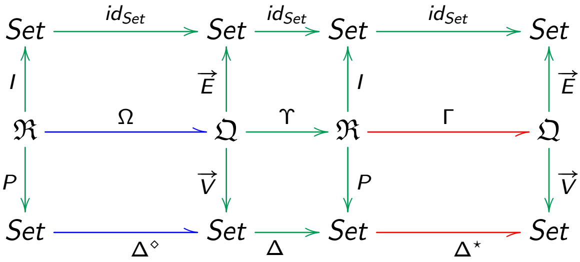


Figure: Natural functor diagram for $\mathfrak{Q}$ & $\mathfrak{R}$

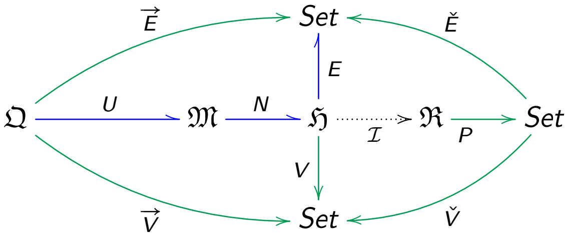


Figure: Functorial diagram for $\mathfrak{Q}$, $\mathfrak{M}$, $\mathfrak{H}$, & $\mathfrak{R}$

G. Chen, V. Liu, E. Robinson, L. J. Rusnak, and K. Wang.
A characterization of oriented hypergraphic laplacian and adjacency matrix coefficients.
ArXiv. 1704.03599 [math.CO], 2017.

E. Robinson, L. J. Rusnak, M. Schmidt, and P. Shroff.
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ArXiv. 1709.04011 [math.CO], 2017.

L.J. Rusnak.
Oriented hypergraphs: Introduction and balance.
Electronic J. Combinatorics, 20(3)(#P48), 2013

Proof of $perm(L_G)$:

- Via weak walks:

$$\mathrm{perm}(L_G) = \sum_{\pi \in S_V} \prod_{v \in V} \sum_{\omega \in \Omega_{1,\pi}} -sgn(\omega(\overrightarrow{P}_1)),$$

where $\Omega_{1,\pi}$ is the set of all incidence preserving maps $\omega : \overrightarrow{P}_1 \to G$ with $\omega(t) = v$ and $\omega(h) = \pi(v)$.

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  where $\Omega_{1,\pi}$ is the set of all incidence preserving maps $\omega : \overrightarrow{P}_1 \to G$ with $\omega(t) = v$ and $\omega(h) = \pi(v)$.

- Distribute, but do not evaluate, the inner sums for all $v \in V$. Sum passes to incidence preserving maps $c : \coprod_{v \in V} \overrightarrow{P}_1 \to G$ with $\omega(t_v) = v$, $\omega(h_v) = \pi(v)$, and $\{\omega(h_v) \mid v \in V\} = V$.

Proof of $perm(L_G)$:

- Via weak walks:

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- Distribute, but do not evaluate, the inner sums for all $v \in V$. Sum passes to incidence preserving maps $c : \coprod_{v \in V} \overrightarrow{P}_1 \to G$ with $\omega(t_v) = v$, $\omega(h_v) = \pi(v)$, and $\{\omega(h_v) \mid v \in V\} = V$.
- Collecting permutomorphic contributors gives:

$$\mathrm{perm}(L_G) = \sum_{\pi \in S_V} \sum_{c \in \mathfrak{C}_\pi(G)} \prod_{v \in V} \sigma(c(i_v))\sigma(c((j_v)).$$

Proof of $perm(L_G)$ continued:

- Consider the product $\prod_{v \in V} \sigma(c(i_v))\sigma(c((j_v))$.

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- Consider the product $\prod_{v \in V} \sigma(c(i_v))\sigma(c((j_v))$.
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- Factor out -1 from every adjacency that is negative in G, producing a factor of $(-1)^{nc(c)}$ and a net factor of $(-1)^{oc(c)+nc(c)}$.

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- Thus,

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- Combine to get

$$\mathrm{perm}(L_G) = \sum_{c \in \mathfrak{C}(G)} (-1)^{oc(c)+nc(c)}.$$