The Fixed Vertex Property, Products and Binary Constraint Satisfaction

Bernd.Schroeder@usm.edu

Fixed Point Properties

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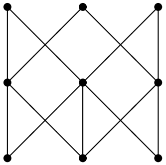

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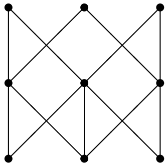

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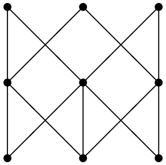

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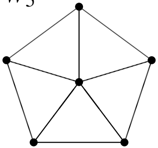

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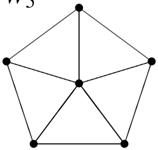

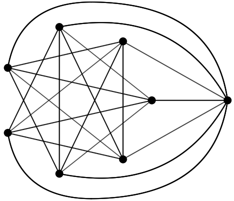

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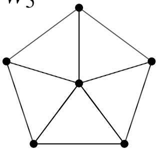

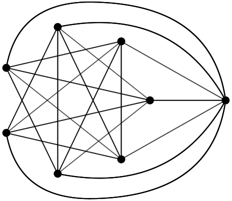

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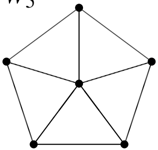

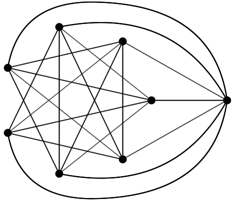

have the fixed vertex property. But why do they?

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(So that's also why we defined all edges as two-element subsets.)

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This means that graph colorings are homomorphisms into complete graphs, which is why the term “generalized coloring” is sometimes associated with the study of graph homomorphisms.

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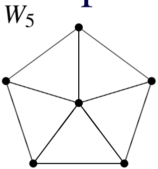

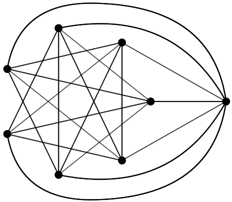

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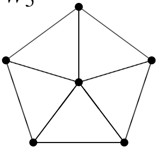

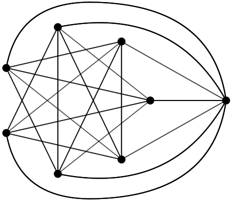

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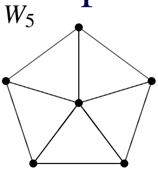

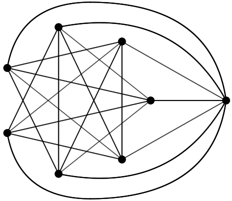

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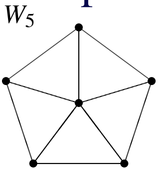

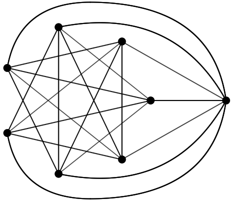

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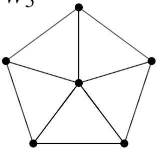

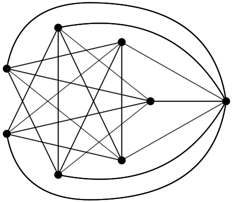

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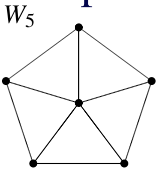

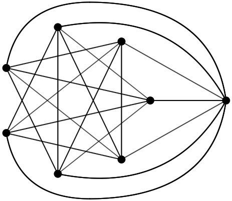

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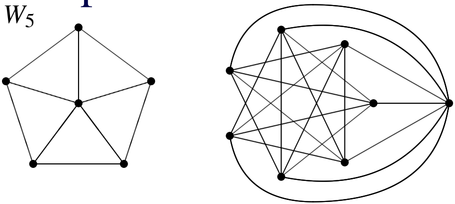

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A dominating vertex does not guarantee the fixed vertex property: Attaching a dominating vertex to the two graphs above produces graphs without the fixed vertex property.

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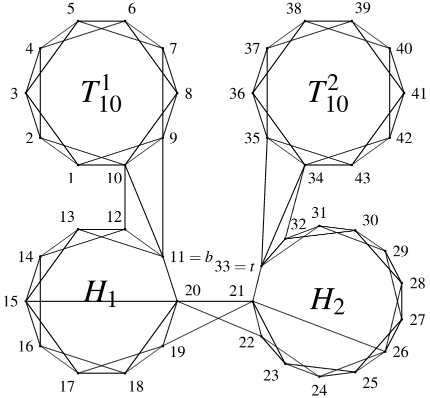

A Semi-Solved Problem

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Certain tools for finite ordered sets are not available for general infinite ordered sets. They are also not available for the fixed vertex property for graphs. So, with the fixed point property (including products of ordered sets with the fixed point property) embedding into the fixed vertex property, maybe an analysis of the fixed vertex property for products of graphs could provide new insights?

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One of them corresponds to the one for ordered sets (with the embedding from the previous panel) and maybe the other ones are interesting, too.

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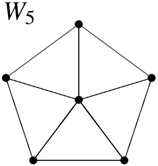

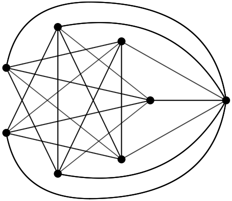

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3. *A set* $\mathscr{C}$ *of unary and binary constraints.*

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1. *A set of variables* $x_1, \cdots, x_r$, *(take the vertices of the graph)*
2. *A set of domains* $D_1, \cdots, D_r$, *one for each variable, (make* $D_i := V \setminus \{x_i\}$*)*
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A **solution** is a set of instantiations $\{(x_1, f_1), \ldots, (x_r, f_r)\}$ such that any subset $\{(x_j, f_j), (x_k, f_k)\}$ is such that $(f_j, f_k) \in C_{jk}$.

(My) Implementation

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Expanded Binary Constraint Network. (Interpret as a graph.)

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- Solutions correspond to r-cliques.

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- Remove any remaining $\{(x_j, f_j), (x_k, f_k)\}$ such that there is an $x_i \notin \{x_j, x_k\}$ for which there is no triangle of the form $\{(x_j, f_j), (x_k, f_k), (x_i, y)\}$

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- There are many more consistency enforcing mechanisms.

(My) Implementation

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Once the front end for direct products was written, the algorithm worked embarrassingly fast ... even faster if we enforce $(2,2)$-consistency, that is, delete all edges $\{(x_j,f_j),(x_k,f_k)\}$ of the network such that there are 2 more variables $x_i,x_\ell\notin\{x_j,x_k\}$ such that the edge is not part of a 4-clique $\{(x_j,f_j),(x_k,f_k),(x_i,y),(x_\ell,v)\}$.

Back to Products

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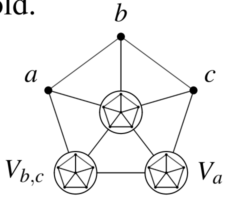

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I have little intuition beyond the fact that an endomorphism of the form $f(x, y) = (g(x), h(y))$ must have a fixed point and that many endomorphisms must be of this form. (See below for cartesian products.)

I think the strong product operation also corresponds to the product operations for the embeddings of ordered sets. (Should be a similar proof as for the direct product, but more technical.)

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Lemma.

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Lemma. *Any triangle $(x_1, y_1) \sim (x_2, y_2) \sim (x_3, y_3) \sim (x_1, y_1)$ in $G \square H$ satisfies either $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$.*

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Lemma. *Any triangle* $(x_1, y_1) \sim (x_2, y_2) \sim (x_3, y_3) \sim (x_1, y_1)$ *in* $G \square H$ *satisfies either* $x_1 = x_2 = x_3$ *or* $y_1 = y_2 = y_3$. *Therefore, if* G *and* H *have the fixed vertex property and any two vertices of* G *are connected by a path such that every edge is contained in a triangle, then* $G \square H$ *has the fixed vertex property.*

... so there is no counterexample here

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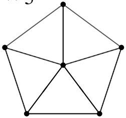
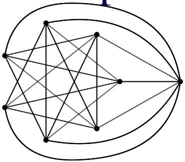
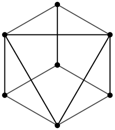
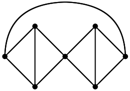
W_5

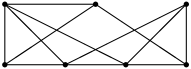




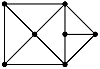
Graph 649 Graph 738 Graph 769

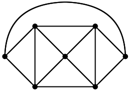


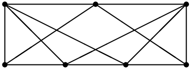




Graph 766 Graph 825 Graph 794







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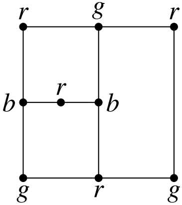

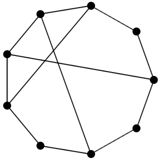

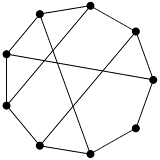

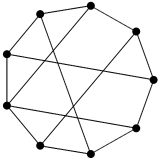

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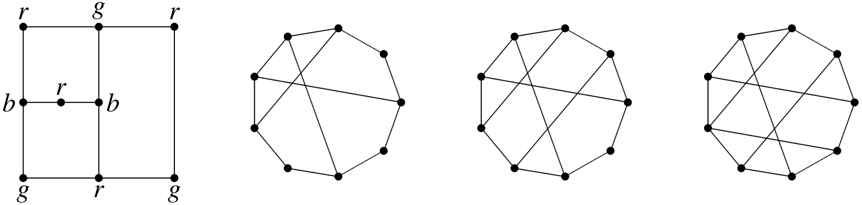

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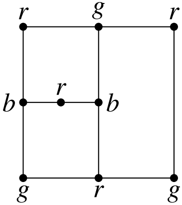

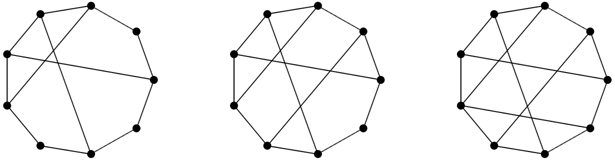

... and it naturally brings up the following question: What kinds of graphs have the fixed vertex property and no triangles?

A Universal Tool for Fixed Points

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Definition.

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Definition. *Let P be a*

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Theorem.

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Theorem. *Let P be*

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Proof.

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Theorem. *Let P be a structure with the fixed point property and let $r : P \to P$ be a retraction. Then $r[P]$ has the fixed point property.*

Proof. Let $f : r[P] \to r[P]$ be structure-preserving.

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Proof. Let $f : r[P] \to r[P]$ be structure-preserving. Then $f \circ r : P \to P$ is structure-preserving, too

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Theorem. *Let P be*

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and let $r : P \to P$ be a retraction. Then $r[P]$ has the fixed point property.

A Universal Tool used for Fixed Vertices

Definition. *Let G be a graph. Then an endomorphism $r : V \to V$ is called a* **retraction** *iff $r^2 = r$ (that is, iff r is* **idempotent***). We will say that $R \subseteq V$ is a* **retract** *of P iff there is a retraction $r : V \to V$ with $r[V] = R$.*

Theorem. *Let G be a graph with the fixed vertex property and let $r : V \to V$ be a retraction. Then $G[r[V]]$ has the fixed vertex property.*

Searching for Graphs without Triangles and with the Fixed Vertex Property

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