The poset on connected graphs is Sperner

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Introduction and Terminology

A partially ordered set (poset $(P, \leq)$) is a binary relation $\leq$ over a set P such that for $a, b, c \in P$.

1. $a \leq a$ (reflexivity).
2. if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry).
3. if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity).

A poset is *graded* if there exists a partition of P into subsets $A_0, \ldots, A_m$ such that A_0 is the set of minimal elements and whenever $x \in A_i$ and $y \in A_j$ with $x < y$, $\nexists z \in P$ s.t. $x < z < y$.

A *chain* $\mathcal{C} \subset P$ is a set of pairwise comparable elements.

An *anti-chain* $\mathcal{A} \subset P$ is a set of pairwise incomparable elements.

Introduction and Terminology

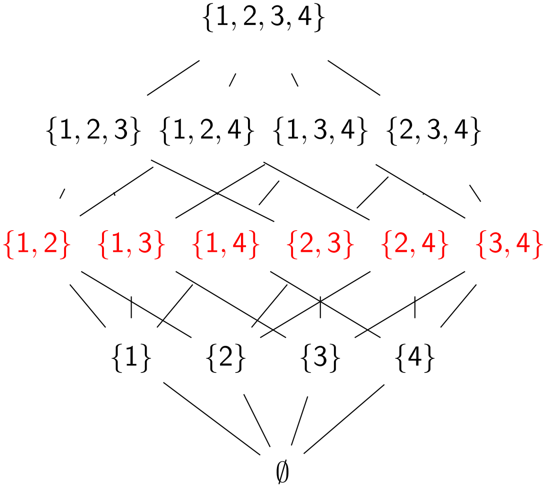
Theorem - Sperner 1928

Let $\mathcal{A}$ be an antichain of subsets of an n-set S.
Then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Definition

A poset P is called *Sperner* if the largest antichain is largest the sized level.

Figure: Poset on [4]



Introduction and Terminology

Let $\mathcal{C}$ be the set of connected graphs on vertex set $[n]$.

Order the graphs in $\mathcal{C}$ with the following natural partial ordering:

If $G, H \in \mathcal{C}$, let $G \leq H$ if $E(G) \subset E(H)$.

Katona - 2014

Is $(\mathcal{C}, \leq)$ Sperner?

Let $m = \binom{n}{2}$, $M = \lceil m/2 \rceil$.

Theorem 1 I. Tomon, S.G.Z.S. - 2017

For sufficiently large n, $(\mathcal{C}, \leq)$ is Sperner and $\mathcal{C}^{(M)}$ is the unique largest antichain.

This result appeared in JCT-A August, 2017.

Introduction and Terminology

Let $X \subset \mathcal{C}^{(k)}$, for $1 \leq k \leq m$

Upper Shadow

The *lower shadow* of X is $\Delta(X) = \{G \in C^{(k-1)} : \exists H \in X, G < H\}$

Lower Shadow

The *upper shadow* of X is $\nabla(X) = \{G \in C^{(k+1)} : \exists H \in X, H < G\}$

Main Tools

Lovász 1979

Let $X \subset \mathcal{C}^{(k)}$ be nonempty, x a real number such that $|X| = \binom{x}{k}$. Then,

$$|\Delta(X)| \geq \binom{x}{k-1}.$$

In particular,

$$\frac{|\Delta(X)|}{|X|} \geq \frac{k}{x-k+1}.$$

Why not Kruskal-Katona?

Comparability Graph

Bipartite graph B with vertex partition $(X, \nabla(X))$, and edges exist between two vertices if they are comparable.

The Easy Part

Matchings between levels

Hall's Theorem

Let $G = (A, B; E)$ be a bipartite graph. There is a complete matching in G from A to B if and only if $|X| \leq |\Gamma(X)|$ for all $X \subset A$, where $\Gamma(X)$ denotes the set of vertices adjacent to some element of X.

Lemma 1

Suppose there is a complete matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k+1)}$ for $k = n-1, \ldots, M-1$ and from $\mathcal{C}^{(l+1)}$ to $\mathcal{C}^{(l)}$ for $l = M, \ldots, m-1$. Then the largest antichain in $\mathcal{C}$ is $\mathcal{C}^{(M)}$.

Using complete matchings, one can build a chain partition of $\mathcal{C}$ into $|\mathcal{C}^{(M)}|$.

The Easy Part

Lemma 2

There is a complete matching from $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k+1)}$ for $k = n-1, \ldots, M-1$.

For $X \subset \mathcal{C}^{(k)}$, apply Hall's theorem on $(X, \nabla(X))$

Lemma 3

There is a complete matching from $\mathcal{C}^{(k+1)}$ to $\mathcal{C}^{(k)}$ for $k = M+n, \ldots, m$.

For $X \subset \mathcal{C}^{(k)}$, apply Hall's theorem on $(X, \Delta(X))$

The More Difficult Part

Matchings between levels $\mathcal{C}^{(k)}$ to $\mathcal{C}^{(k-1)}$ for $M+1 \leq k < M+n$ is more difficult due to disconnectedness.

Question 1

How many edges can a graph G have, whose removal destroys the connectivity?

Question 2

What is the number of 2-edge-connected graphs G on $[n]$ in which there are exactly r edges, whose removal destroys the 2-edge-connectivity?

Settling the Difficulties

Need to show for all $X \subset \mathcal{C}^{(k)}$, that $|\Delta(X)| \geq |X|$

1. Break $|X|$ into 2-edge-connected and non-2-edge-connected sets
2. Observe that if the two sets are not both roughly the same size, the larger of the two has a shadow $> |X|$.

Let G be a 2-edge-connected graph with a set $R(G)$ edges such that $G - e$ is not 2-edge-connected, for $e \in R(G)$.

Lemma 4

Let A be the set of 2-edge-connected graphs such that $R(G) = r$ on level k. Let ϵ be a positive real. Then there exists $n_1(\epsilon)$ such that if $n > n_1(\epsilon)$, the following holds. For positive integers r and k with $2 \leq r \leq n$, and $M \leq k \leq M + n$. Then,

$$|A| \leq \binom{\binom{n-r/2}{2}+\epsilon rn}{k}$$

Conclusion

There is a complete matching for $k = n - 1, \ldots, M - 1$ by Lemma 2.

There is a complete matching for $k = M + n, \ldots, m$ by Lemma 3.

By Lemma 4, we are able to see that for $M + 1 \leq k < M + n$, we can ensure that there aren't too many disconnected graphs.

Open Questions

Let G be a connected graph and let $C'(G)$ be the family of subgraphs of G that are connected on $[n]$. Define the partial ordering $<$ on $C'(G)$ as usual: $H < H'$ if $E(H) \subset E(H')$.

Open Question 1

Is $(C'(G), <)$ Sperner?

Open Questions

Let GP be a monotone graph property (a family of graphs closed under isomorphism, and adding edges) and let GP_n denote the family of graphs in GP with vertex set $[n]$. Also, for $k = 0, \ldots, \binom{n}{2}$ let $GP^{(k)}$ be the set of graphs in GP with k edges. Define the partial order $<$ as usual.

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or which graph properties GP is it true that the largest antichain in $(GP_n, <)$ is $GP_n^{(k)}$ for some k?

Questions

Questions?