On Automorphisms of Haar graphs of Abelian Groups

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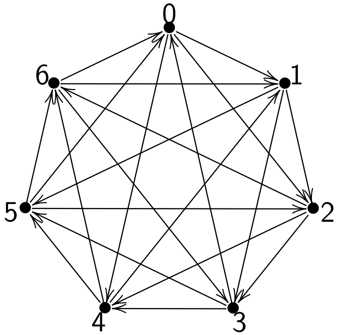

Figure: The Cayley digraph $\mathrm{Cay}(\mathbb{Z}_7, \{1, 2, 4\})$

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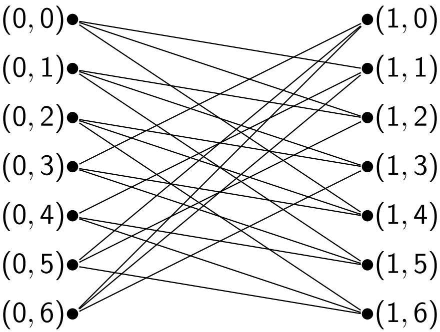

Figure: The Heawood graph as $\mathrm{Haar}(\mathbb{Z}_7, \{1, 2, 4\})$

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- QUESTION: Is there a correspondence between the automorphism group of $\mathrm{Cay}(G, S)$ and the automorphism group of $\mathrm{Haar}(G, S)$?

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2. Γ *is connected, and the action of* $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ *is unfaithful on* B_1. *There exists a subgroup* $1 < B \leq A$ *such that* $\Gamma \cong \mathrm{Haar}(A/B, S) \wr \bar{K}_\beta$ *where* $\beta = |B|$ *and* S *is interpreted as a set of cosets of* B *in* A, *and* $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathrm{Haar}(A/B, S)) \wr S_\beta$,

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4. *the action of* $\mathrm{fix}_{\mathrm{Aut}(\Gamma)}(\mathcal{B})$ *on* B_1 *is faithful but the actions on* B_0 *and* B_1 *are not equivalent permutation groups.*

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Definition

Let Γ be a digraph, and $s \geq 1$. A sequence of arcs $a_1, \ldots, a_s \in A(\Gamma)$ is an **alternating s-arc** *if there exists vertices $x_0, \ldots, x_s \in V(\Gamma)$, $x_i \neq x_{i+2}$, and for $1 \leq m \leq s$ the arc $a_m = (x_{m-1}, x_m)$ if m is odd while $a_m = (x_m, x_{m-1})$ if m is even.*

So a natural approach to classify s-arc-transitive graphs is to determine all of the s-arc-transitive graphs which DON'T have intransitive normal subgroups (so-called "base" graphs), and then consider all possible choices of N to reconstruct all s-arc-transitive graphs. While theoretically possible, this is not in practice doable.

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Theorem

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This result theoretically gives a way of determining all s-arc-transitive Cayley graphs of generalized dihedral groups, a fairly large class of graphs. All s-arc-transitive Cayley graphs of cyclic and dihedral groups are known.

Thanks!