Unbreakable Frame Matroids

Tara Fife, Dillon Mayhew, James Oxley, and Charles Semple

Mississippi Discrete Math Workshop

What is a Matroid?

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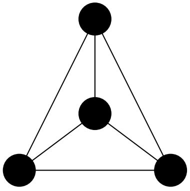
- A *circuit* is $C \in \mathscr{C}$ if C is a minimal dependent set.
- A matroid is *connected* if every two elements are in a common circuit.
- The *rank*, $r(X)$, of a set X is the size of the largest independent subset of X.

Graphic Matroids



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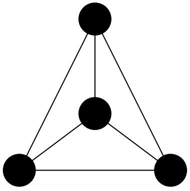

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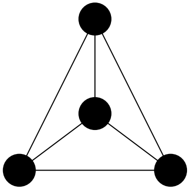

The *circuits* of $M(G)$ are the cycles of G.

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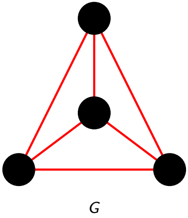
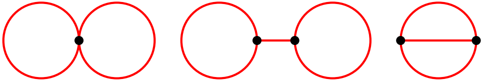
The *rank* of X is the number of vertices spanned by X minus the number of components of X.

Bicircular Matroids



The *circuit* of $B(G)$ are the subdivisions of Θ-graphs, tight handcuffs, and loose handcuffs.



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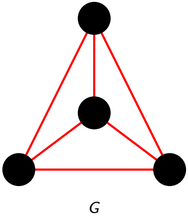

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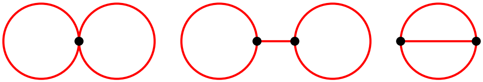

The matroid $B(G)$ might be connected when G has a cut vertex, and might be disconnected even when G is 2-connected.

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Rule: If C_1 and C_2 are both in Ψ and $C_1 \cap C_2$ is a path, then the third cycle in $C_1 \cup C_2$ is also in Psi.

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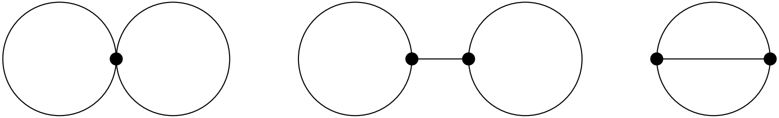

The circuits of $M(G, \Psi)$ are elements of Ψ together with Θ-graphs and handcuffs of G using only unbalanced cycles.

Rank of Frame Matroids

- If $M = M(G)$, then $r(M)$ is $|V(G)|$ minus the number of connected components of G.

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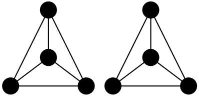
- If $M = M(G, \Psi)$, then $r(M)$ is $|V(G)|$ minus the number of balanced components of G.

Recognising when $M(G, \Psi)$ is disconnected.

1. 2 components



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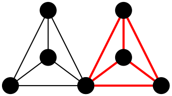

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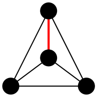
3. C is unBalanced iff $e \in C$.



4. Unbalanced loops at a vertex.

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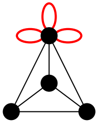

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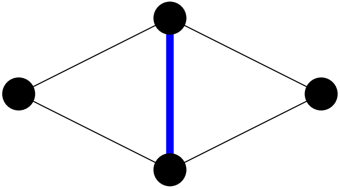

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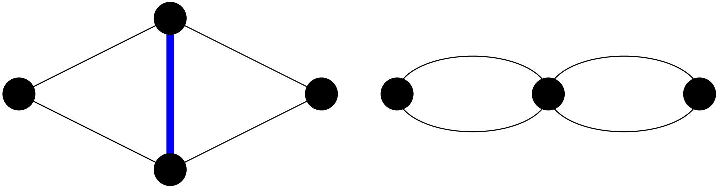

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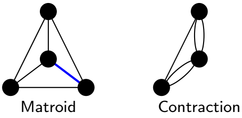

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We say that a connected matroid M is *unbreakable* if for every subset F of $E(M)$, the matroid M/F is connected, except possibly with loops.

Lemma (Pfeil)

Let M be a loopless matroid, then M is unbreakable iff $si(M)$ is unbreakable.

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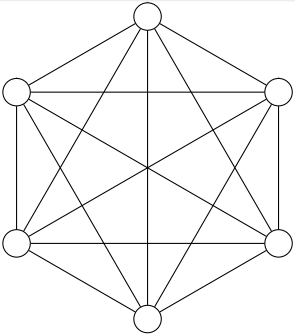

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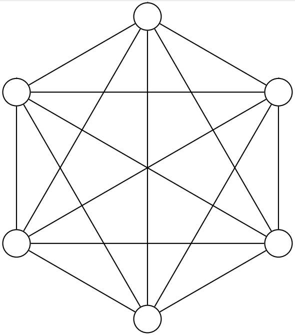

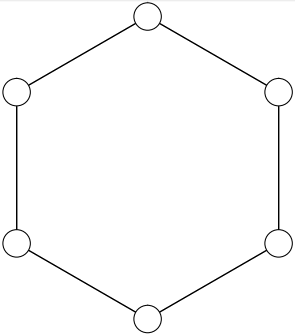

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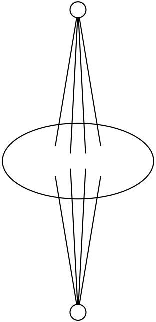

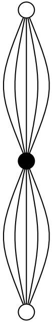

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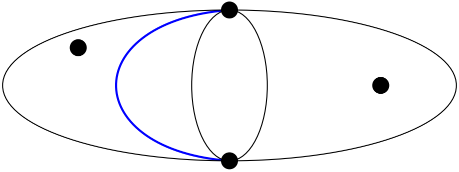

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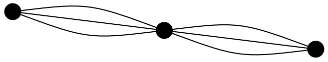

Understanding Unbalanced Frame Matroids

We only need to consider the case where M is 3-connected.

Theorem (Pfeil)

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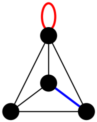
- When M is graphic, we understand what's going on.
- If G is not 3-connected then $|V(G)|$ is small.
- If G is 3-connected, then $si(G)$ is almost a complete graph.

Contraction and Deletion



Matroid

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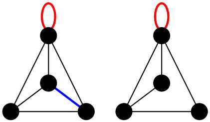

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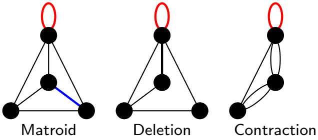

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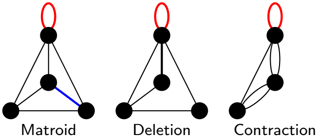

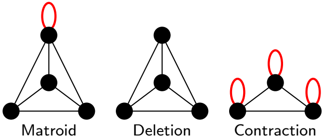

Magic Theorem

Lemma

Suppose that u *and* v *are non-adjacent vertices of* G. *Then* $G - \{u, v\}$ *is balanced.*

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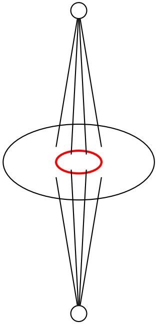

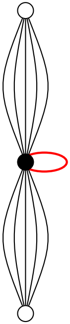

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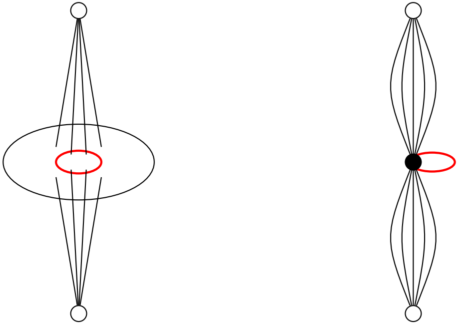


Main Result

Theorem

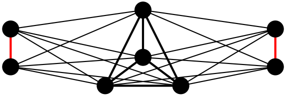
Let $M(G,\Psi)$ is a 3-connected unbreakable Frame matroid. If G has no isolated verticies and $|V(G)| \geq 7$, then $si(G)$ has at most 2-non-edges.

Question (Peter Nelson)

Suppose we have a large complete graph H, and we arbitrarially remove at most k edges. How big can we make k to guarantee that there is some graph G, with $si(G) = H$, and collection Ψ, so that $M(G,\Psi)$ is a 3-connected unbreakable matroid?

Thank You!

An Example



Matroid

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$$\lambda_M(X) = r(X) + r(E - X) - r(M)$$

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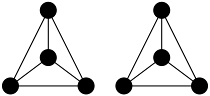

G has two components

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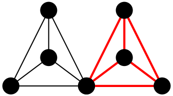

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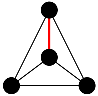

C is unbalanced iff $e \in C$

If M has no 1- or 2-sepperations, then M is 3-connected.

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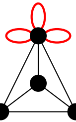

All Cycles are Balanced loops

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Higher Connectivity Continued

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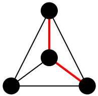

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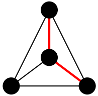

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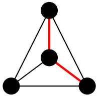

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$\lambda(\{e,f\}) = 2 + 3 - 4 = 1.$ $\quad$ So $\{e,f\}$ is a 2-seperator.

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Corollary

If C is an unbalanced cycle, then $G - C$ is complete.

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Assume that G has a 2-seperation, and show that $|V(G)|$ is small.

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