Improving Olson's Generalization of the Erdős-Ginzburg-Ziv Theorem

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Subsequence Sums

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Let G be an abelian group. A sequence

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$$\Sigma_n(S) = \{g \in G : \quad g \text{ is the sum of an } n\text{-term subsequence of S}\}$$

The Erdős-Ginzburg-Ziv Theorem

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Let S be a sequence of terms from a finite abelian group G. Then

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  (Gao, 1995) Optimal bound

  $$|S| \geq |G| + \mathsf{D}(G) - 1,$$

  where $\mathsf{D}(G)$ is the Davenport Constant of G, i.e, the minimal integer such that

  $$|S| \geq \mathsf{D}(G) \quad \Rightarrow \quad 0 \in \Sigma(S) := \bigcup_{n \geq 1} \Sigma_n(S).$$

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  If all terms of S from same coset $\alpha + H$, then $\Sigma_{|G|}(S) \subseteq H$.

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The Coset Condition

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We say the sequence S of terms from the finite abelian group G satisfies the **coset condition** if, for every proper subgroup $H < G$ and $\alpha \in G$, there are at least $|G/H| - 1$ terms of S lying outside the coset $\alpha + H$.

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- Equivalently, at most $|S| - |G/H| + 1$ terms of S from the coset $\alpha + H$.
- Special case: $H = \{0\}$ is trivial, then coset condition ensures that the maximum multiplicity in S is at most $|S| - |G| + 1$.

Olson's Generalization

Theorem (Olson (1977))

Let S be a sequence of terms from a finite abelian group G. Suppose S satisfies the coset condition. Then

$$|S| \geq 2|G| - 1 \quad \Rightarrow \quad \Sigma_{|G|}(S) = G.$$

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Question: Is the bound $|S| \geq 2|G| - 1$ tight?

Improvements to Olson's Result

- (Gao 1995) Suppose S satisfies the coset condition. Then

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- Basic bounds:

$$1 + \sum_{i=1}^{r}(m_i - 1) = \mathrm{D}^*(G) \leq \mathrm{D}(G) \leq |G|,$$

where $G \cong \mathbb{Z}/m_1\mathbb{Z} \times \ldots \times \mathbb{Z}/m_r\mathbb{Z}$ with $m_1 \mid \ldots \mid m_r = \exp(G)$.

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- (G., Marchan, Ordaz (2009)) Suppose S satisfies the coset condition. Then

$$|S| \geq |G| + \mathsf{D}^*(G) - 1 \quad \Rightarrow \quad \Sigma_{|G|}(S) = G.$$

An important observation

- If $|S| = |G| + n$, then $\Sigma_{|G|}(S) = \sigma(S) - \Sigma_n(S)$.

$$\begin{array}{ccc} T & \longleftrightarrow & S \cdot T^{[-1]} \\ \sigma(T) & \longleftrightarrow & \sigma(S \cdot T^{[-1]}) = \sigma(S) - \sigma(T) \end{array}$$

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- Natural to impose max multiplicity instead at most

$$|S| - |G| = n.$$

Tight Bounds

Theorem (G. (2018))

Let G be a finite abelian group, let $n \geq 2$, and let S be a sequence of terms from G with

(a) $|S| = |G| + n$,

(b) *satisfying the coset condition, and*

(c) *having maximum multiplicity at most n.*

Then $\Sigma_{|G|}(S) = G$ whenever

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Then $\Sigma_{|G|}(S) = G$ whenever

1. $n \geq \exp(G)$, *or*
2. $n \geq \exp(G) - 1$, $G \cong H \oplus C_{\exp(G)}$, *and either* $|H|$ *or* $\exp(G)$ *is prime, or*
3. $n \geq \frac{|G|}{p} - 1$ *and G is cyclic, where p is the smallest prime divisor of* $|G|$, *or*
4. $n \geq 2$ *and either* $\exp(G) \leq 3$, *or* $|G| < 12$, *or* $\exp(G) = 4$ *and* $|G| = 16$.

Tight Bounds

Suppose S satisfies the coset condition. Then

$$|S| \geq |G| + \exp(G) \quad \Rightarrow \quad \Sigma_{|G|}(S) = G,$$

with improvements for near cyclic groups that make the resulting bounds optimal.

The Partition Theorem

Theorem (Subsums Kneser Theorem)

Let G be an abelian group, let S be a sequence with

$$|S| \geq n \geq 1 \text{ and maximum multiplicity at most } n.$$

Let $H = \mathsf{H}(\Sigma_n(S)) = \{g \in G : \; g + \Sigma_n(S) = \Sigma_n(S)\}$.

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Then

$$|\Sigma_n(S)| \geq |S| - (n-1)|H| + e(|H| - 1) + \rho,$$

where $X \subseteq G/H$ is the subset of all $x \in G/H$ having multiplicity at least n in $\varphi_H(S)$,

$e \geq 0$ is the number of terms from $\varphi_H(S)$ not contained in X, and

$\rho = |X||H|n + e - |S| \geq 0$.

Consequences of the Partition Theorem

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- Estimates for the number of $|G|$-term zero-sums in a sequence of length n. For $n \leq \frac{19}{3}|G|$, there are at least $\binom{\lceil \frac{n}{2} \rceil}{|G|} + \binom{\lfloor \frac{n}{2} \rfloor}{|G|}$.

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- Common tool used for handling zero-sum Ramsey Theory questions
- Connection with Matroid Theory:

A Matroid Theory Conjecture

Conjecture (Schrijver and Seymour)

Let M be a matroid, let G be an abelian group, and let $w : M \to G$ be a function. If

$$w(M) = \{\sum_{x \in B} w(x) : \ B \textit{ is a basis for } M\}$$

is aperiodic, then

$$|w(M)| \geq \sum_{x \in G} \mathrm{rk}\Big(w^{-1}(x)\Big) - \mathrm{rk}(M) + 1$$

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is aperiodic, then

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- To derive the previous result, take M to be $|S|$ points in $\mathbb{R}^n$ in general position, each mapped to one of the terms from S.

Improved Version for large n

Theorem

Let G be a finite abelian group, let $n \geq 1$, let S be a sequence of terms from G, and let $H = \mathsf{H}(\Sigma_n(S))$. Suppose $n \geq \exp(G/H) + 1$.

Then the terms of S can be partitioned into nonempty sets $A_1, \ldots, A_n \subseteq G$ such that either

(i) $|\Sigma_n(S)| \geq \min\{|G|, |S| - n + 1\}$, *or else*

(ii) *there exists a nontrivial subgroup $K \leq H < G$ such that*

  (a) $\Sigma_n(S) = \sum_{i=1}^{n} A_i$,

  (b) *the coset condition fails for both H and K, and*

  (c) $\sum_{i \in I} A_i$ *is a K-coset, for some $I \subseteq [1, n]$.*

Summary

Suppose S satisfies the coset condition with maximum multiplicity at most n. Then

$$n \geq \exp(G) + 1 \quad \Rightarrow \quad |\Sigma_n(S)| \geq \min\{|G|, |S| - n + 1\},$$

with improvements for near cyclic groups that make the resulting bounds optimal.

Sumsets

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For subsets A, $B \subseteq G$, we let

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- If $|A + B| < |A| + |B|$, then Kemperman (1960) gave a complete recursive description of all possible A, $B \subseteq G$ for an arbitrary abelian group G.
- Extended to the case $|A + B| \leq |A| + |B|$ (G. 2005).

An iterated extension

Theorem (G. 2018)

Let G be a nontrivial abelian group, let $A \subseteq G$ be a finite subset with $\langle A - A \rangle = G$, and let $n \geq 3$ be an integer. Suppose nA is aperiodic, $|A| \geq 4$ and $|nA| < (|A|+1)n - 3$. Then one of the following must hold.

(i) *There is an arithmetic progression $P \subseteq G$ such that $A \subseteq P$ and $|P| \leq |A| + 1$.*

(ii) *There are subgroups $K_1, K_2, H < G$ with $H = K_1 \oplus K_2 \cong C_2 \oplus C_2$ such that*

$$z+A = \Big(x+K_1\Big)\cup\Big(y+H\Big)\cup\ldots\cup\Big((r-1)y+H\Big)\cup\Big(ry+K_2\Big) \quad \text{with } r \geq 1,$$

for some $z \in G$, $x \in H$ and $y \in G \setminus H$.

(iii) *There is a subgroup $H < G$ with $|H| = 2$ such that*

$$z + A = \{x\} \cup \Big(y + H\Big) \cup \ldots \cup \Big(ry + H\Big) \cup \{(r+1)y\} \quad \text{with } r \geq 1,$$

for some $z \in G$, $x \in H$ and $y \in G \setminus H$.

An iterated extension

(iv) There is a nontrivial subgroup $H < G$ such that

$$z+A = \{0\} \cup \Big(y + (H \setminus \{x\})\Big) \cup \Big(2y + H\Big) \cup \ldots \cup \Big(ry + H\Big) \quad \text{with } r \geq 1,$$

for some $z \in G$, $x \in H$ and $y \in G \setminus H$, with $r \geq 2$ when $|H| = 2$.

(v) There is a nontrivial subgroup $H < G$, nonempty $A_0 \subseteq H$ and set

$$P = A_0 \cup (y + H) \cup \ldots \cup (ry + H) \quad \text{with } r \geq 1,$$

for some $y \in G \setminus H$, such that

(a) $A_0 \subseteq z + A \subseteq P$ with $|P| = |A| + \epsilon \leq |A| + 1$, for some $z \in G$,
(b) nA_0 is aperiodic,
(c) either $|A_0| = 1$ or $|nA_0| < \min\{|\langle A_0 \rangle_*|, (|A_0| + 1 - \epsilon)n - 3\}$,
(d) $nA \setminus nA_0$ is H-periodic, and
(e) $|nA| - |A|n = |nA_0| - |A_0|n + \epsilon n$.

Thanks!