Single-Distance Graphs on $\mathbb{Q}^n$

Joint work with Sheng Bau

For $X \subseteq \mathbb{R}^n$, $d > 0$, $G(X, d)$ is the graph with vertex set X, with
$xy \in E(G) \Longleftrightarrow |x-y| = d$
($| \ | =$ Euclidean norm on $\mathbb{R}^n$).

When P is a graph parameter, like χ or ω, we abbreviate
$P(G(X,d))$ to $P(X, d)$

The vast savannah of Euclidean coloring problems grew from one question, posed by (18-year-old) Edward Nelson in 1950:

What's $\chi(\mathbb{R}^2, 1)$?

Nelson and Isbell, unpublished, 1950:
$\chi(\mathbb{R}^2, 1) \in \{4, 5, 6, 7\}$.
(Later rediscovered by Hadwiger and the Moser brothers)

For the full story: The Mathematical Coloring Book, by Alexander Soifer.

Breaking (or recently broken) news: July, 2018 issue of <u>Geombinatorics</u> contains The Chromatic Number of the Plane Is At Least 5, by Aubrey D. N. J. de Grey.

First mention in print of Nelson's question appeared in Martin Gardner's <u>Mathematical Games</u> column in <u>Scientific American</u>, October, 1960. [He heard about it from one of the Mosers.] There ensued a slow crescendo of activity, peaking in the 1970's, dying down to a steady popping frequency by the late 1990's. Not all about $\chi(\mathbb{R}^n, 1)$, but mostly.

1973, Douglas Woodall: $\chi(\mathbb{Q}^2, 1) = 2$

1979, Benda and Perles (unpublished manuscript; finally published in ~~2000~~ 2000):

$\chi(\mathbb{Q}^3, 1) = 2$, $\chi(\mathbb{Q}^4, 1) = 4$. Also, $G(\mathbb{Q}^n, 1)$ is not connected, $1 \le n \le 4$.

1981, Frankl and Wilson: $\chi(\mathbb{Q}^n, 1) \ge (1 + o(1)) r^n$ for some $r > 1$. [Largest such r found so far, 1.173, proved by Raigorodskij in 2001.]

Why is it always about the distance 1?

Clearly $G(\mathbb{R}^n, d) \simeq G(\mathbb{R}^n, 1)$ for all $d > 0$ and positive integers n, but the same is not true with $\mathbb{R}^n$ replaced by $\mathbb{Q}^n$. Obviously, if $d > 0$ is not realized as a distance in $\mathbb{Q}^n$, then $G(\mathbb{Q}^n, d) \not\simeq G(\mathbb{Q}^n, 1)$. Less trivially:

$(0,0,0)$, $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ induce a K_4 in $G(\mathbb{Q}^3, \sqrt{2})$, so

$$\chi(\mathbb{Q}^3, \sqrt{2}) \geq 4 > 2 = \chi(\mathbb{Q}^3, 1)$$

[this was first noticed by Benda and Perles, I think], so $G(\mathbb{Q}^3, \sqrt{2})$ and $G(\mathbb{Q}^3, 1)$ are non-trivial and not isomorphic.

Some discoveries about graphs $G(\mathbb{Q}^n, d)$, when d is realized as a distance in $\mathbb{Q}^n$:

1989 PJ: let $D = \{\sqrt{\tfrac{p}{q}} \mid p$ and q are odd positive integers$\}$.

$\chi(\mathbb{Q}^n, D) = 2$ if $n \in \{1, 2, 3\}$, and

$\chi(\mathbb{Q}^4, D) = 4$.

Let $B_1(\mathbb{Q}^n) = \max[\chi(\mathbb{Q}^n, d) ; d > 0]$

2001 A. Abrams + PJ : $B_1(\mathbb{Q}^2) = 2$.

2008 PJ : $B_1(\mathbb{Q}^3) = B_1(\mathbb{Q}^4) = 4$

Both of these are easy consequences of our main results, later. The 2001 result was blown away by the following discovery by Matt Noble (with proof by PJ)

Theorem If $d > 0$ is a distance realized ~~by~~ in $\mathbb{Q}^2$, then $G(\mathbb{Q}^2, d) \simeq G(\mathbb{Q}^2, 1)$

Proof d realized in $\mathbb{Q}^2 \Rightarrow d = \sqrt{a^2+b^2}$ for some $a, b \in \mathbb{Q}$, not both 0.

Let $z = a + ib \in \mathbb{Q}[i] \simeq \mathbb{Q}^2$ (as a v.s.)

Multiplication by $\frac{1}{z}$ maps $\mathbb{Q}[i]$ 1-1 onto itself and takes points a distance d apart into points a distance 1 apart. $\boxtimes$

Main results.

(1) If n is an even positive integer and $d = \sqrt{a^2+b^2} > 0$, $a, b \in \mathbb{Q}$, is realized as a distance in $\mathbb{Q}^n$, then $G(\mathbb{Q}^n, 1) \cong G(\mathbb{Q}^n, d)$.

(2) If $4 \mid n$ and $d > 0$ is realized as a distance in $\mathbb{Q}^n$ then $G(\mathbb{Q}^n, d) \cong G(\mathbb{Q}^n, 1)$.

Problems

i) Is 3 the only value of n such that $\chi(\mathbb{Q}^n, 1) < B_1(\mathbb{Q}^n)$?

ii) How many isomorphism classes are there for the ~~B~~ graphs $G(\mathbb{Q}^n, d)$, $n \in \{3, 5, 6, 7\}$?
[Remark: Except for the empty graph on $\mathbb{Q}^n$, every isomorphism class of graphs $G(\mathbb{Q}^n, d)$ contains a graph $G(\mathbb{Q}^n, \sqrt{m})$, m positive and square-free.]

iii) What are the clique numbers $\omega(\mathbb{Q}^n, 1)$?
[Can show that $\max_{d>0} \omega(\mathbb{Q}^n, d) = \max_{d>0} \omega(\mathbb{Z}^n, d)$, and the latter values are <u>known</u>: Schoenberg, 1937.
If, for some n, $\omega(\mathbb{Q}^n, 1) < \max_{d} \omega(\mathbb{Z}^n, d)$, then for some d realized in $\mathbb{Q}^n$, $G(\mathbb{Q}^n, 1) \not\cong G(\mathbb{Q}^n, d)$.]