A q-analogue and a symmetric function analogue of a result by Carlitz, Scoville and Vaughan

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Carlitz, Scoville, and Vaughan's result

Put $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!n!}$ and $\frac{1}{f(z)} = \sum_{n=0}^{\infty} \omega_n \frac{z^n}{n!n!}$.

It follows that $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \omega_k = 0$.

The Bessel function $J_0(z)$ is essentially $f(z^2)$ and ω_k's are the the coefficients of the reciprocal Bessel function.

Given $\sigma \in \mathcal{S}_n$, a permutation of $[n] = 1, 2, ..., n$, a number $i \in [n-1]$ is called an *ascent* of σ if $\sigma(i) < \sigma(i+1)$.

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C-S-V result

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- For example, $\omega_2 = 3$: $(12, 21)$, $(21, 12)$, $(21, 21)$.
- Carlitz, Scoville and Vaughan's result provided a combinatorial interpretation of the coefficient ω_k in the reciprocal Bessel function.

Segre Product Poset

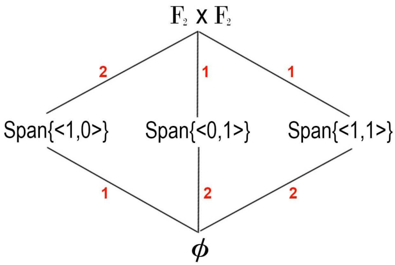
Definition of Segre Products of Posets

Let $f : P \longrightarrow S$ and $g : Q \longrightarrow S$ be poset maps. Let $P \circ_{f,g} Q$ be the induced subposet of the product poset $P \times Q$ consisting of the pairs $(p, q) \in P \times Q$ such that $f(p) = g(q)$. Let $S = \mathbb{N}$. When P is a pure poset with rank function $f = \rho$, $P \circ_{\rho,g} Q$ is the *Segre product of P and Q* with respect to g, and we denote it by $P \circ_g Q$.

- Let B_n be the subset lattice ordered by inclusion. The q-analogue poset of B_n is $B_n(q)$, which consists of all subspaces of an n-dimensional vector space over $\mathbb{F}_q$, ordered by inclusion.
- Segre product poset $B_n \circ_\rho B_n$: $\{(a, b) \in B_n \times B_n \text{ and } \rho(a) = \rho(b)\}$
- Segre product poset $B_n(q) \circ_\rho B_n(q)$

The subspace lattice $B_2(2)$

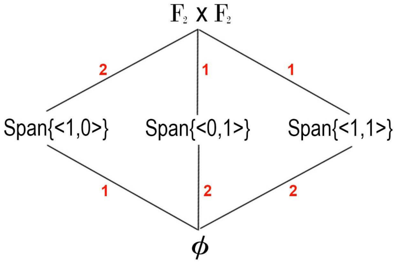
- An *edge labeling* is a map $\lambda : \mathcal{E}(P) \longrightarrow \Lambda$, where Λ is a poset.
- A labeling of $B_2(2)$:



- A maximal chain c is then associated with a word $\lambda(c)$. A chain c is *increasing* if $\lambda(c)$ is strictly increasing and *decreasing* if $\lambda(c)$ is weakly decreasing.
- The left most chain of $B_2(2)$ is increasing with a label 12 and the other two chains are decreasing.

The subspace lattice $B_2(2)$

- An EL-labeling of $B_2(2)$:



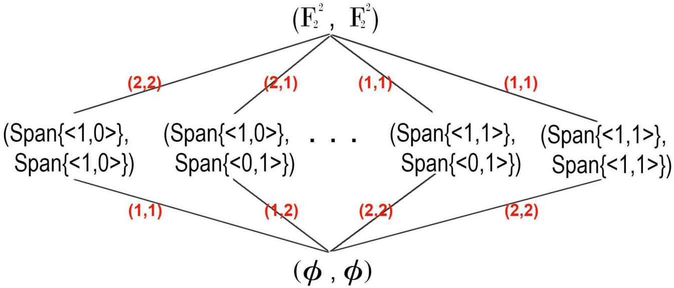
Definition of EL-Labeling

An edge labeling λ of a poset P is called an *EL-labeling* (Edge Lexicographical) if for every interval $[x, y]$ in P,

1. there is a unique increasing maximal chain c in $[x, y]$, and
2. the word $\lambda(c)$ lexicographically precedes $\lambda(c')$ for all other maximal chains c' in $[x, y]$.

The Segre product poset $B_2(2) \circ_\rho B_2(2)$

- An EL-labeling of $B_2(2) \circ_\rho B_2(2)$:



- The left most chain of $B_2(2) \circ_\rho B_2(2)$ is increasing with a label $(12, 12)$.
- The labels of decreasing chains in $B_n(q) \circ_\rho B_n(q)$ are pairs of permutations $(\sigma, \omega) \in \mathcal{S}_n \times \mathcal{S}_n$ with no common ascent. Denote the set of all such pairs by $\mathcal{D}_n$. e.g. The second chain has label $(12, 21)$.

A few more definitions

- $[n]_q = q^{n-1} + q^{n-2} + ... + 1$ is the q-analogue of the natural number n
- $[n]_q! = [n]_q[n-1]_q...[2]_q[1]_q$
- The q-analogue of $\binom{n}{k}$ is $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$.
- For a permutation $\sigma \in \mathcal{S}_n$, the *inversion statistic* is defined by

$$inv(\sigma) := |\{(i,j) : 1 \leq i < j \leq n \text{ and } \sigma(i) > \sigma(j)\}|.$$

e.g. $inv(312) = |\{(1,2),(1,3)\}| = 2$

q-analogue to C-S-V result

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Notation: Let $\mathcal{D}_n$ be the set of all pairs of permutations $(\sigma, \omega) \in \mathcal{S}_n \times \mathcal{S}_n$ with no common ascent.

Theorem 1 (L.): a q-analogue to C-S-V result

Let $P_n(q)$ be the proper part of the Segre product poset $B_n(q) \circ_\rho B_n(q)$. Let $W_n(q)$ be the total number of decreasing maximal chains of $P_n(q)$. Then

$$\sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q^2 W_i(q) = 0$$

and $W_i(q) = \sum_{(\sigma,\omega)\in\mathcal{D}_n} q^{inv(\sigma)+inv(\omega)}$.

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$$\sum_{i=0}^{n} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q^2 W_i(q) = 0 \qquad (1)$$

and $W_i(q) = \sum_{(\sigma,\omega)\in\mathcal{D}_n} q^{inv(\sigma)+inv(\omega)}$.

- Let $F(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{[n]_q![n]_q!}$. The function $F\left(\left(\frac{z}{2(1-q)}\right)^2\right)$ is the q-Bessel function $J_0^{(1)}(z;q)$.
- Equation (1) implies that $W_n(q)$ satisfies $\frac{1}{F(z)} = \sum_{n=0}^{\infty} W_n(q) \frac{z^n}{[n]_q![n]_q!}$.

q-analogue to C-S-V result

- In fact, $W_n(q)$ is the Euler characteristic of $B_n(q) \circ_\rho B_n(q)$ up to a sign.

The coefficients $W_n(q)$ of the reciprocal q-Bessel function is the absolute value of Euler characteristic of $B_n(q) \circ_\rho B_n(q)$, which counts the number of decreasing maximal chains of the Segre product poset $B_n(q) \circ_\rho B_n(q)$.

- We can get C-S-V's result by letting $q = 1$. Their work included general cases where occurrences of common ascent are allowed. Our proof provides a less technical approach by utilizing Björner and Wachs' work on shellability and poset homology.

A well-known symmetric function identity

For $n \geq 1$, $\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0$.

Symmetric functions can be used to describe representations of $\mathcal{S}_n$.

(Frobenius) characteristic map

Let $\mathcal{R}^n$ be the space of class functions on $\mathcal{S}_n$ and Λ^n denote the space of degree n symmetric functions. The *(Frobenius) characteristic map* ch^n: $\mathcal{R}^n \longrightarrow \Lambda^n$ is defined by

$$ch^n(\chi) = \sum_{\mu \vdash n} z_\mu^{-1} \chi_\mu p_\mu,$$

where $z_\mu = \prod i^{m_i} m_i!$ for a partition $\mu = (1^{m_1} 2^{m_2} \ldots)$ and χ_μ is the value of χ on the class μ.

A well-known symmetric function identity

For $n \geq 1$,

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0. \qquad (2)$$

Notations: $\mathcal{R} := \oplus_n \mathcal{R}^n$, $\Lambda := \oplus_n \Lambda^n$ is the ring of symmetric functions, and $ch := \oplus_n ch^n$, $ch : \mathcal{R} \longrightarrow \Lambda$, is a homomorphism of rings.

Let $\bar{B}_i$ be the proper part of B_i. The elementary symmetric function e_i is $ch(H_{i-2}(\bar{B}_i))$, the (Frobenius) characteristic of the representation of $\mathcal{S}_i$ on the top homology of $\bar{B}_i$.

Our analogue to equation (2) will give the $\mathcal{S}_n \times \mathcal{S}_n$ version of this identity.

The product Frobenius characteristic map

We defined the product Frobenius characteristic map to help understand representations on $\mathcal{S}_n \times \mathcal{S}_n$. Consider two sets of variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$.

Definition

Let χ be a class function on $\mathcal{S}_m \times \mathcal{S}_n$. The *product Frobenius characteristic map* $ch : \mathcal{R} \times \mathcal{R} \longrightarrow \Lambda(x) \times \Lambda(y)$ is defined as:

$$ch(\chi) = \sum_{(\mu,\lambda) \vdash (m,n)} z_\mu^{-1} z_\lambda^{-1} \chi_{(\mu,\lambda)} p_\mu(x) p_\lambda(y),$$

where $\chi(\mu, \lambda)$ is the value of χ on the class (μ, λ).

The class (μ, λ) is indexed by a partition μ of m and a partition λ of n that tell us the cycle shapes of elements of $\mathcal{S}_m$ and $\mathcal{S}_n$ respectively.

Our analogue to $\sum_{i=0}^{n}(-1)^i e_i h_{n-i} = 0$

- Just like the usual characteristic map, the product Frobenius characteristic map is a homomorphism of rings as well.

Theorem 2 (L.): a symmetric function analogue

For the subset lattice B_n with rank function ρ, let P_n be the proper part of the Segre product poset $B_n \circ_\rho B_n$. The action of $\mathcal{S}_n \times \mathcal{S}_n$ induces a representation on the reduced top homology of P_n. Let $ch(\widetilde{H}_{n-2}(P_n))$ be the product Frobenius characteristic of this representation. Then

$$\sum_{i=0}^{n}(-1)^i ch(\widetilde{H}_{i-2}(P_i))h_{n-i}(x)h_{n-i}(y) = 0. \qquad (3)$$

Stable principal specialization

The product Frobenius characteristic $ch(\widetilde{H}_{n-2}(P_n))$ is a symmetric function in two sets of variables $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$.

Definition of $ps(f)$

The *stable principal specialization* $ps : \Lambda \longrightarrow \mathbb{Q}[q]$ of a symmetric function f is defined by

$$ps(f) = f(1, q, q^2, ...).$$

To find $ps(ch(\widetilde{H}_{n-2}(P_n)))$, we take the stable principal specialization of the function in each set of variables, that is substituting $(1, q, q^2, ...)$ for both $(x_1, x_2, ...)$ and $(y_1, y_2, ...)$.

The connection between two analogues

By taking the stable principal specialization of the symmetric function analogue, we can recover the polynomial identity $\sum_{i=0}^{n}(-1)^i \begin{bmatrix} n \\ i \end{bmatrix}_q^2 W_i(q) = 0$ in our q-analogue and arrive at the following result.

Theorem 3 (L.)

Let P_n be the proper part of the Segre product poset $B_n \circ_\rho B_n$. Then

$$ps(ch(\widetilde{H}_{n-2}(P_n))) = \frac{W_n(q)}{\prod_{i=1}^{n}(1-q^i)^2},$$

where $ch(V)$ is the product Frobenius characteristic of a $\mathcal{S}_m \times \mathcal{S}_n$-module V.

Remark: The Euler characteristic $W_n(q)$ of $B_n(q) \circ_\rho B_n(q)$ can be obtained from specializing the product Frobenius characteristic of the representation of $\mathcal{S}_n \times \mathcal{S}_n$ on top homology of $B_n \circ_\rho B_n$.

Questions

1. Let P_n be a poset that has an $\mathcal{S}_n$ action. Let $P_n(q)$ be a q-analogue of P_n. What properties does P_n need so that the Euler characteristic of $P_n(q)$ can be obtained from specializing the Frobenius characteristic of the representation of $\mathcal{S}_n$ on the homology of P_n.
2. In our symmetric function analogue, P_i denotes the proper part of $B_i \circ_\rho B_i$. Can $ch(\widetilde{H}_{i-2}(P_i))$ be written explicitly in symmetric functions?
3. Is there any general setting in which our results hold? Can we find similar results on posets other than B_n and $B_n(q)$?
4. Can we find some results for representations of $\mathcal{S}_n \times \mathcal{S}_n \times \mathcal{S}_n$ on the homology of $B_n \circ B_n \circ B_n$?