Recognizing when bicoset graphs are X-joins.

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Definition
Let G be a group, let L and R be subgroups of G, and let S be a union of double cosets of R and L in G, namely, $S = \bigcup_i Rs_iL$. Define a bipartite graph $\Gamma = \mathrm{B}(G, L, R, S)$ with bipartition $V(\Gamma) = G/L \cup G/R$ and edge set $E(\Gamma) = \{\{gL, gsR\} : g \in G, s \in S\}$. This graph is called the **bi-coset graph** with respect to L, R, and S. We call S the **connection set** of Γ.

Bicoset and Haar Graphs

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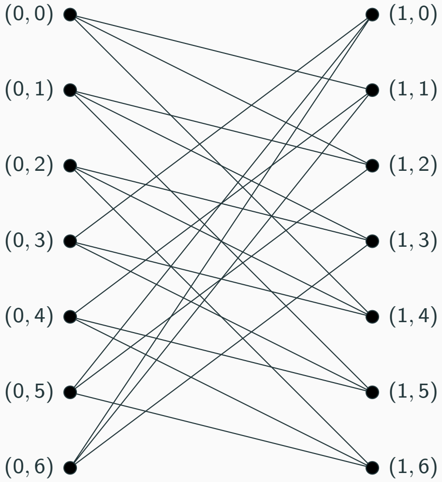
Definition
Let G be a group and $S \subseteq G$. Define the **Haar graph**, denoted $\mathrm{Haar}(G, S)$ with connection set S to be the graph with vertex set $\mathbb{Z}_2 \times G$ and edge set $\{(\{0, g), (1, gs)\} : g \in G,\ s \in S\}$.

Bicoset and Haar Graphs



Haar($\mathbb{Z}_7, \{1, 2, 4\}$) AKA The Heawood Graph

Join-Partition

Definition
Let Γ be a bi-coset graph $\mathrm{B}(G, H_0, H_1, S)$ where the left partition B_0 consists of the left cosets of H_0 and the right partition B_1 consists of the left cosets of H_1. Let $H_i \leq K_i \leq G$, $i = 0, 1$. Define the **join-partition of $V(\Gamma)$ with respect to K_0 and K_1**, denoted $\mathcal{P}(K_0, K_1)$, of the vertices of Γ as follows:

1. Let $\mathcal{P}_i$ be the partition of B_i that consists of the left cosets of K_i in G. Note $\mathcal{P}_i$ is a block system of G with its action on B_i by left multiplication, $i = 0, 1$.
2. The partition $\mathcal{P}(K_0, K_1)$ of $V(\Gamma)$ is $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$. This partition of the vertices of Γ does not necessarily form a block system of $\mathrm{Aut}(\Gamma)$ as Γ may not be vertex-transitive.

Join-Partition

$\mathcal{P}$ is a refinement of the natural partition $\mathcal{B}$, where $\mathcal{B}$ partitions $V(\Gamma)$ into B_0 and B_1.

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Lemma

Let $\Gamma = B(G, H_0, H_1, S)$ and $\mathcal{P}$ a partition of $V(\Gamma)$ that refines $\mathcal{B}$. Then $\mathcal{P}$ is a G-invariant partition of $V(\Gamma)$ under the left multiplication action of G if and only if there exists $H_0 \leq K_0 \leq G$ and $H_1 \leq K_1 \leq G$ such that $\mathcal{P}$ is the (K_0, K_1)-join partition of $V(\Gamma)$.

X-joins

Definition

Let X be a graph, $Y = \{Y_x : x \in X\}$ a collection of graphs indexed by $V(X)$. By the ***X*-join** of Y is meant the graph $Z = \bigvee(X, Y)$ with vertex set

$$V(Z) = \{(x, y) : x \in X, y \in Y_x\}$$

and edge set

$$E(Z) = \{\{(x, y), (x', y')\} : \{x, x'\} \in E(X) \text{ or } x = x' \text{ and } \{y, y'\} \in E(Y_x)\}.$$

X-joins

To construct the X-join of Y:

1. Replacing each vertex of X by the graph $Y_x \in Y$.
2. Insert either all or none of the possible edges between vertices of Y_u and Y_v depending on whether or not there is an edge between u and v in X.

If the Y_x's are all isomorphic, then the X-join of $\{Y_x : x \in X\}$ is the **wreath product** $X \wr Y$, where $Y \cong Y_x$ for all $x \in X$.

X-joins

Example
Let $X = K_2$, the complete graph on 2 vertices, and let $Y = \{\bar{K}_2, K_3\}$.

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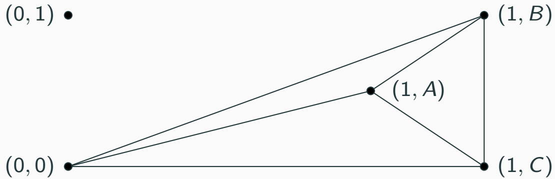

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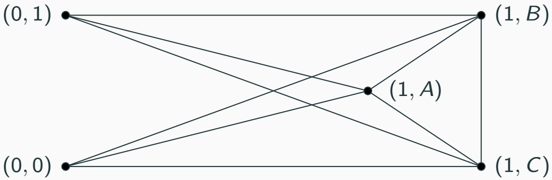


Quotient Graph

Definition
Let Ω be a set, and $\mathcal{P}$ a partition of Ω. Let Γ be a digraph with vertex set Ω. Define the **quotient digraph** of Γ with respect to $\mathcal{P}$, denoted $\Gamma/\mathcal{P}$, by $V(\Gamma/\mathcal{P}) = \mathcal{P}$ and $(P_1, P_2) \in A(\Gamma/\mathcal{P})$ if and only if $(p_1, p_2) \in A(\Gamma)$ for some $p_1 \in P_1$ and $p_2 \in P_2$.

When bicoset graphs are X-joins

Theorem
Let G be a group, $H_0 \leq K_0 \leq G$, $H_1 \leq K_1 \leq G$, $m_0 = [K_0 : H_0]$, and $m_1 = [K_1 : H_1]$. Let $S \subseteq G$ such that S is a union of (H_0, H_1)-double cosets in G, and $\Gamma = B(G, H_0, H_1, S)$. Let $X = \Gamma/\mathcal{P}$ where $\mathcal{P}$ is the join-partition of Γ with respect to K_0 and K_1, $Y_{g,i}$ be the empty graph on the left cosets of H_i contained in gK_i, and $Y = \{Y_{g,i} : g \in G, i \in \mathbb{Z}_2\}$. Then Γ is the X-join of Y if and only if whenever $P_0 \in \mathcal{P}_0$ and $P_1 \in \mathcal{P}_1$, then there is an edge $\{x_0, x_1\}$ from a vertex $x_0 \in P_0$ to a vertex $x_1 \in P_1$ if and only if every edge of the form $\{x_0, x_1\}$ with $x_0 \in P_0$ and $x_1 \in P_1$ is contained in $E(\Gamma)$.

Remark: This theorem allows us to be able to recognize X-joins with complements of complete graphs from a graph theoretic point of view.

When bicoset graphs are X-joins

Theorem

Let $\Gamma = B(G, H_0, H_1, S)$ *be a connected bi-coset graph,* $H_i \leq K_i \leq G$*,* $i = 0, 1$*, and* $\mathcal{P} = \mathcal{P}(K_0, K_1)$ *be the join-partition of* $V(\Gamma)$ *with respect to* K_0 *and* K_1*. Let* $X = \Gamma/\mathcal{P}$*. For* $gK_i \in \mathcal{P}$*, let* $Y_{g,i}$ *the empty graph with vertex set* gK_i*, and let* $Y = \{Y_{g,i} : g \in G, i \in \mathbb{Z}_2\}$*. Then* Γ *is the* X*-join of* Y *if and only if* S *is a union of* (K_0, K_1)*-double cosets in* G*. If such a* $K_0, K_1 \leq G$ *exists, then*

$$B(G, H_0, H_1, S) = \bigvee(\Gamma/\mathcal{P}, Y) \cong \bigvee(B(G/L, K_0/L, K_1/L, T), Y)$$

where $L = core_G(K_0) \cap core_G(K_1)$*, and* $T = \bigcup_{s \in S} (K_0/L)(sL)(K_1/L)$*.*

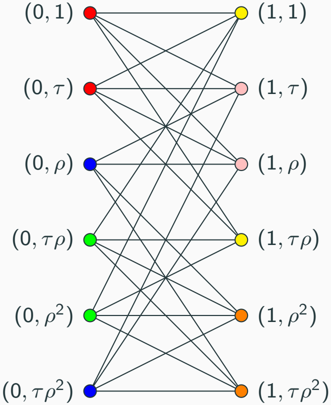
Remark: This theorem helps of identify when bicoset graphs are X-joins of empty graphs by looking at the connection set S. And, we identify what the X-join is in terms of another bicoset graph.

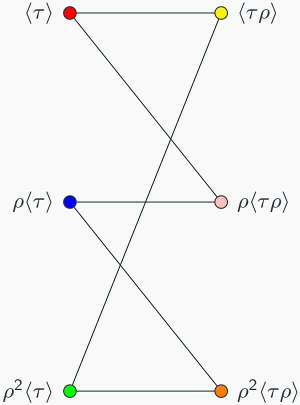
Example

Example

Let $\Gamma = \mathsf{Haar}(D_6, \{1, \tau, \rho, \tau\rho\})$, where D_6 is the dihedral group with six elements. Note that S is exactly the double coset $\langle\tau\rangle\tau\langle\tau\rho\rangle$. Then by the previous theorem we know that Γ is isomorphic to an X-join of empty graphs, in this case $Y = \{\bar{K}_2, \bar{K}_2, \bar{K}_2, \bar{K}_2, \bar{K}_2, \bar{K}_2\}$ as the order of each coset is two. Thus, Γ is in fact a wreath product.

$\mathbf{Haar}(D_6, \{1, \tau, \rho, \tau\rho\}) = \mathbf{B}(D_6, \langle\tau\rangle, \langle\tau\rho\rangle, T) \wr \bar{K}_2$





Next Steps

To do next: Finish up the results about the automorphism group.
Question: When is a disconnected bicoset graph an X-join of graphs?