Variations on a Theme of Turán

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VCU
College of Humanities
and Sciences

Part 1: Introduction / History

Who?

Joint work with almost everyone...

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Jozsef Balogh (Illinois)
Mauricio Collares Neto (IMPA)
Andrzej Czygrinow (ASU)
Nathan Kettle (Cambridge / IMPA / $$$)
Hong Liu (Illinois)
Rob Morris (IMPA)
Maryam Sharifzadeh (Illinois)
Jangwon Yie (ASU)

The setup...

Fix a graph H

The Forbidden Subgraph Problem

The setup...

Fix a graph H (small)

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Fix a graph H (small), and consider an arbitrary order n graph G

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The Extremal Question:

Given a graph H, how many edges can an n-vertex H-free graph contain?

A Little More Formal

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We'll use H_{Ex} to represent some $H \in \text{Ex}(n, H)$.

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Classical Theorems

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$$\mathrm{ex}(n, K_{r+1}) = |E(T(n, r))| \leq \left(1 - \frac{1}{r}\right) \binom{n}{2}.$$

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Let $r = r(n) \in \mathbb{N}_0$ be a function satisfying $r \leq (\log n)^{1/4}$ for every $n \in \mathbb{N}$. Then almost all K_{r+1}-free graphs on n vertices are r-partite.

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Further directions...

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Further directions...

- What happens if r grows faster?
- Can we do similar things forbidding other growing families of graphs?

Classical Theorems (cont.)

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Note:

Erdős-Stone gives very little information about forbidding bipartite graphs!

Part 2: Multiple Copies

A Generalization

A New Question:

What if we allow a few copies of H, but not more?

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Slightly More Formal:

How many edges can an n-vertex graph contain, given that it doesn't contain k vertex disjoint copies of H?

Some More Notation

Definition

For $k \in \mathbb{N}$ and a graph H, we use $k \cdot H$ to denote k vertex disjoint copies of H.

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For graphs G, H, we use $G + H$ to denote the join of G and H; that is,

$$V(G + H) = V(G) \cup V(H)$$

$$E(G + H) = E(G) \cup E(H) \cup (V(G) \times V(H))$$

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So for graphs of chromatic number at least 3, the extremal numbers for multiple copies do not change (asymptotically). But what about bipartite graphs?

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A Simple Construction

For any $H_{\mathrm{Ex}} \in \mathrm{Ex}(n-k+1, H)$, $K_{k-1} + H_{\mathrm{Ex}}$ is a $k \cdot H$-free graph on n vertices.

But where is this construction extremal?

Gorgol, 2011

Let P_ℓ denote the path on ℓ vertices, and M_s denote the (nearly) perfect matching on s vertices. Then for $k = 2, 3$ and n sufficiently large,

$$\mathrm{ex}(n, k \cdot P_3) = \binom{k-1}{2} + (k-1)(n-k+1) + \left\lfloor \frac{n-k+1}{2} \right\rfloor,$$

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$$\mathrm{ex}(n, k \cdot P_\ell) = \binom{k \lfloor \frac{\ell}{2} \rfloor - 1}{2} + (k \left\lfloor \frac{\ell}{2} \right\rfloor - 1)(n - k \left\lfloor \frac{\ell}{2} \right\rfloor + 1) + 1_{\ell \text{ is odd}}.$$

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Here, the extremal graph is $K_{k\lfloor \frac{\ell}{2} \rfloor - 1} + E_{n-k\lfloor \frac{\ell}{2} \rfloor + 1}$ (with a single edge added if ℓ is odd), and this is not $K_{k-1} + H_{\mathrm{Ex}}$!

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A graph H is *forestable* if it meets the following conditions:

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2. H contains a cycle,
3. There is a vertex $v \in V(H)$ such that $H[V(H) \setminus v]$ is a forest.

Forestable Graphs

B.-Kettle

For a forestable graph H, $k \in \mathbb{N}$, and n sufficiently large,

$$\mathrm{ex}(n, k \cdot H) = \binom{k-1}{2} + (k-1)(n-k+1) + \mathrm{ex}(n-k+1, H).$$

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Further, every extremal graph is of the form $K_{k-1} + H_{\mathrm{Ex}}$ for some $H_{\mathrm{Ex}} \in \mathrm{Ex}(n-k+1, H)$.

Future Directions

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Big Question

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Part 3: Rainbow Turán Numbers

What if I like coloring?!

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Then,

$$\text{ex}(n, H) = \max\{\|G\| : G \text{ is an } n \text{ vertex } H\text{-saturated graph}\}.$$

Let's add those colors...

Definition

Given an edge coloring $\chi' : E(G) \to [k]$, we say that a copy $H \subseteq G$ is **rainbow** if $\chi'(e) \neq \chi'(f)$ for any $e, f \in E(H)$.

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Definition

G is H-rainbow-saturated if there is a proper edge coloring of G which is rainbow-H-free, but for every $e \notin E(G)$ we have that every proper edge coloring of $G + e$ contains a rainbow copy of H.

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Now we're ready...

Then as before, we can define the rainbow Turán number:

$$\mathrm{ex}^*(n, H) = \max\{\|G\| : G \text{ is an } n \text{ vertex } H\text{-rainbow-saturated graph}\}$$

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Studied sporadically, and then studied in depth by Keevash, Mubayi, Sudakov and Verstraëte (2007).

Some results...

- $\text{ex}^*(n, H) \geq \text{ex}(n, H)$

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- $\mathrm{ex}^*(n, H) \geq \mathrm{ex}(n, H)$ (if you have no copies of H, then you have no rainbow copy of H).

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- $\mathrm{ex}^*(n, H) \geq \mathrm{ex}(n, H)$ (if you have no copies of H, then you have no rainbow copy of H).
- $\mathrm{ex}^*(n, H) = (1 + o(1))\,\mathrm{ex}(n, H)$, whenever $\chi(H) \geq 3$. (KMSV07)

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- $\mathrm{ex}^*(n,H) = (1+o(1))\,\mathrm{ex}(n,H)$, whenever $\chi(H) \geq 3$. (KMSV07)
- So, what about bipartite graphs? (again!)

Degenerate rainbows...

A few results exist...

- $\mathrm{ex}^*(n, K_{s,t}) = O(n^{1/s})$.

$^*k$ edges, $k+1$ vertices

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- $\mathrm{ex}^*(n, C_6) = \Theta(n^{4/3})$.

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- $\mathrm{ex}^*(n, C_6) = \Theta(n^{4/3})$. (KMSV07 – matches non-rainbow order of magnitude, but different constant!!)
- $\frac{k}{2}n \leq \mathrm{ex}^*(n, P_{k+1}) \leq \left\lfloor \frac{3k-1}{2} \right\rfloor n.^*$ (Johnston, Palmer, Sarkar '17)

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- $\frac{k}{2}n \leq \mathrm{ex}^*(n, P_{k+1}) \leq \left\lfloor \frac{3k-1}{2} \right\rfloor n$.* (Johnston, Palmer, Sarkar '17)
- Improved to $\mathrm{ex}^*(n, P_{k+1}) < \left(\frac{9k}{7} + 2\right) n$ (Ergemlidze, Győri, Methuku '18)

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- $\mathrm{ex}^*(n, K_{s,t}) = O(n^{1/s})$. (KMSV07, same as non-rainbow upper bound!)
- $\mathrm{ex}^*(n, C_{2k}) = \Omega(n^{1+1/k})$. (KMSV07, conjectured to be correct order; related to a problem in additive number theory, and likely hard)
- $\mathrm{ex}^*(n, C_6) = \Theta(n^{4/3})$. (KMSV07 – matches non-rainbow order of magnitude, but different constant!!)
- $\frac{k}{2}n \leq \mathrm{ex}^*(n, P_{k+1}) \leq \left\lfloor \frac{3k-1}{2} \right\rfloor n.^*$ (Johnston, Palmer, Sarkar '17)
- Improved to $\mathrm{ex}^*(n, P_{k+1}) < \left(\frac{9k}{7} + 2\right) n$ (Ergemlidze, Győri, Methuku '18)
- Known exactly for forests of stars, and P_4...

$^*k$ edges, $k+1$ vertices

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- **DO MATH, HAVE FUN!**

VCU is actively looking for graduate students in Discrete Math!
http://math.vcu.edu/

- Ghidewon Abay-Asmeron (topological GT)
- Moa Apagodu (enumerative/algebraic comb.)
- Neal Bushaw (extremal/probablistic comb. and GT)
- David Chan (discrete dynamical systems)
- Dan Cranston (graph coloring, structural GT)
- Richard Hammack (algebraic GT)
- Glenn Hurlbert (extremal set theory, comb., GT)
- Craig Larson (automated conjecturing, GT)
- Dewey Taylor (GT, algebraic techniques)

THANK YOU!!