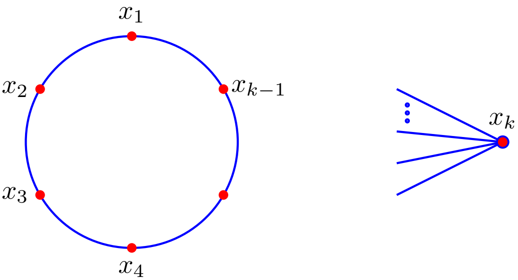
Cycle Traversability and 4-Vertex Linkages for Graphs on Surfaces

— Based on joint work with E. Győri, M. Plummer, C. Stephens and X. Zha

Dong Ye

MIDDLE TENNESSEE STATE UNIVERSITY

(Dirac, 1963). Every k-connected graph has a cycle through any given k vertices.



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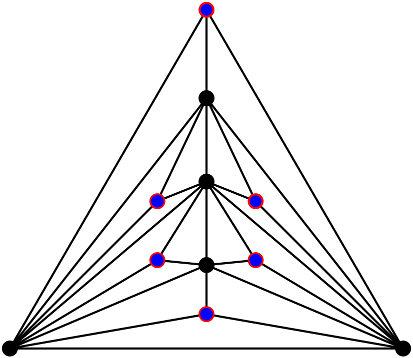
Not all 3-connected plane graphs have a cycle through any given 6 vertices!



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(Kawahabayasi and Ozeki, 2011+). Every 4-connected toroidal triangulation is Hamiltonian.

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Problem (Plummer). Does every polyhedral map have a cycle through any given 5 vertices?

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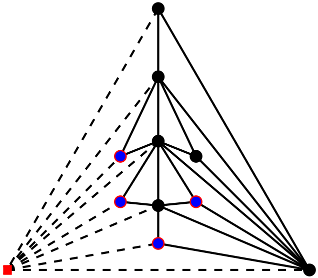

(Győri & Plummer, 2001). For any given $k+1$ vertices with $k \leq 4$, if G has a cycle through k of these given vertices and avoid the remaining one, then G has a cycle through any $k+1$ vertices.

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Problem. For any integer $2 \leq k \leq 6$, what is the largest value $c(k)$ such that every k-connected polyhedral map has a cycle through any given $c(k)$ vertices?

Can we ask a stronger property for polyhedral maps — a cycle through any given 5 vertices in a given order?

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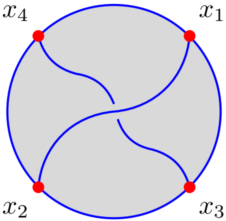
— NO!



A graph G is k-ordered if for any given k vertices $x_1, x_2, \ldots, x_k$, there is a cycle through these vertices in any given order.

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For a given graph H, define $f(H)$ to be the minimum integer k such that every k-connected graph is H-linked.

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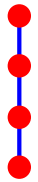
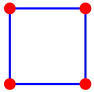
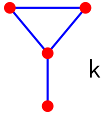


P_4



C_4



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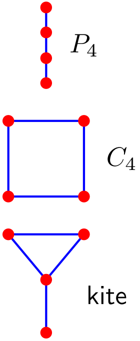

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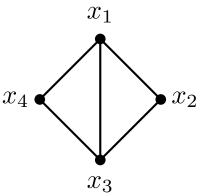
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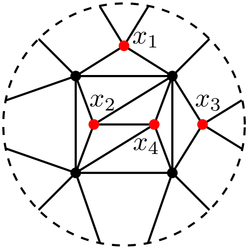
x_4
x_2
x_3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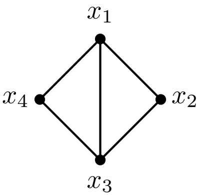

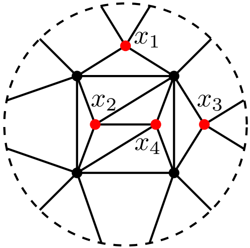

(Stephens, Y. & Zha, 2018+). Every 5-connected surface triangulation is K_4^--linked.