Matroids with a Cyclic Arrangement of Circuits and Cocircuits

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Mathematical Research Institute (MATRIX)
Tutte Centenary Retreat

Mississippi Discrete Mathematics Workshop, Oct., 2019

What are Geometric Presentations?

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- Two dots on a point.

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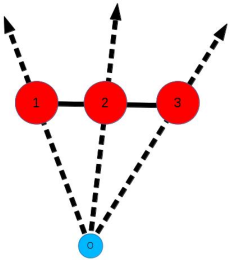
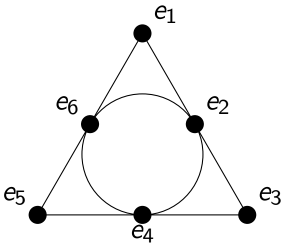


What are Geometric Presentations?

The following are minimally dependent sets.

- Two dots on a point.
- Three (not co-pointer) dots on a line.
- Four (not co-linear) dots on a plane.
- Five (not co-planer) dots in space.
- etc.

What is a Matroid?



Circuits: minimal dependent sets

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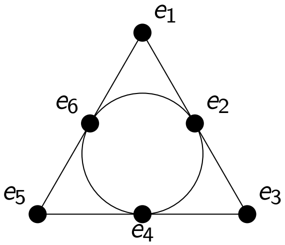

Circuits: minimal dependent sets

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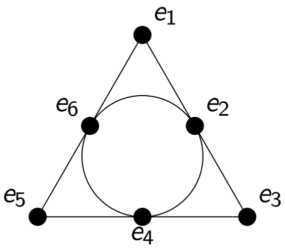

Hyperplanes: set H such that $r(H \cup e) = r(M)$ for all $e \in E(M) - H$ but $r(H) = r(M) - 1$.

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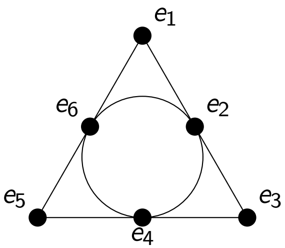

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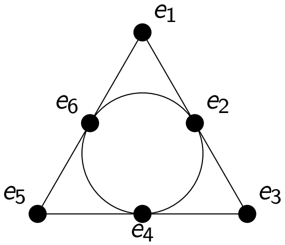

Cocircuits: The complement of a hyperplane, or a minimal set whose removal decreases the rank of the matroid.

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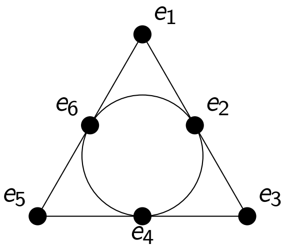

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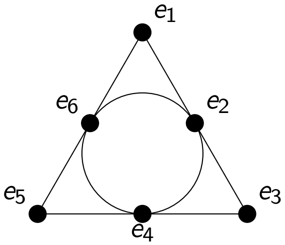

Basis: a maximal independent set.

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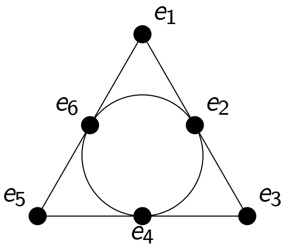

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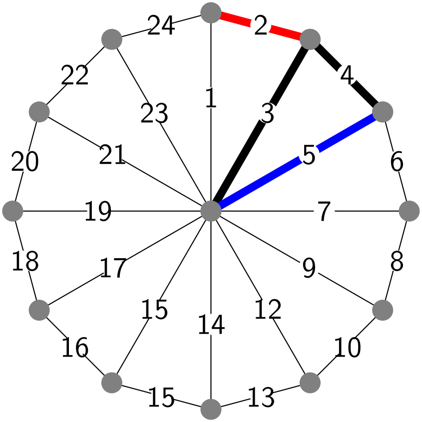
etc.

Wheels and Whirls

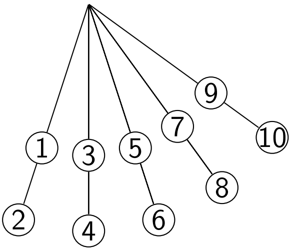


Theorem (Wheels and Whirls Theorem (Tutte))

Let M be a non-empty 3-connected matroid. Then every element of M is in a 3-circuit and a 3-cocircuit if and only if M has rank at least three and is isomorphic to a wheel or a whirl.

Spikes and Swirls

For $r \geq 3$, a rank r *spike* is a matroid on $2r$ elements, where $E(M) = L_1 \sqcup L_2 \sqcup L_2 \sqcup \ldots \sqcup L_r$ and each $L_i \cup L_j$ is a 4-circuit and 4-cocircuit.



Spikes and Swirls

A rank $r \geq 3$ *swirl* is constructed as follows.

- Take a basis $\{b_1, b_2, b_3, \ldots, b_r\}$.
- Add 2-element independent sets $\{e_i, f_i\}$ such that $\{e_i, f_i\} \subseteq \mathrm{cl}(b_i, b_{i+1})$.
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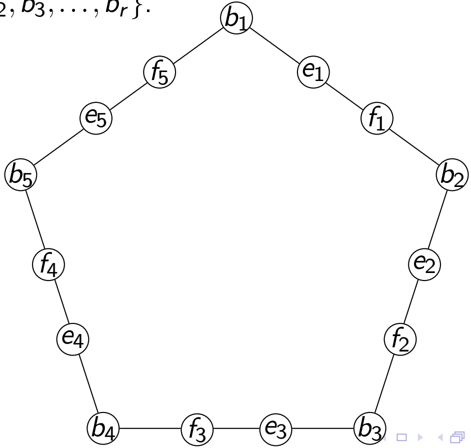

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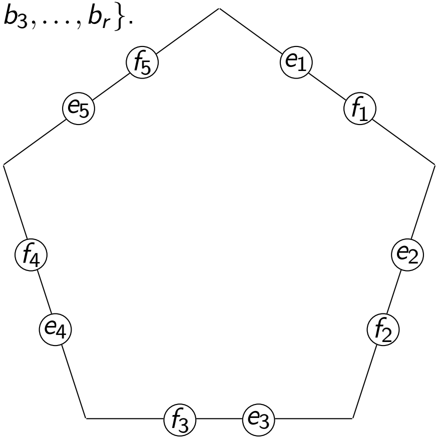

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Cyclic $(t-1, t)$-property

M has the *cyclic $(t-1, t)$-property* if there is a cyclic ordering σ of $E(M)$ such that every $t-1$ consecutive elements of σ is contained in a t-element circuit and a t-element cocircuit.

- A direct sum of copies of $M(C_2)$ is $(1, 2)$-cyclic.

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- Spikes and swirls are $(3,4)$-cyclic.

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- A direct sum of copies of $M(C_2)$ is $(1,2)$-cyclic.
- Wheels and whirls are $(2,3)$-cyclic.
- Spikes and swirls are $(3,4)$-cyclic.
- Motivating Question: Are these the only $(3,4)$-cyclic matroids?

Notation

M is a $(t-1,t)$-cyclic matroid of size n.

- $X_i = \{i, i+1, \ldots, i+t-2\}$
  a $t-1$ interval starting at i
- C_i is a fixed circuit containing X_i
- C_i^* is a fixed cocircuit containing X_i

Main Result

Theorem (Preview)

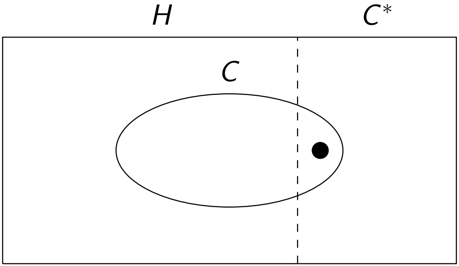
Let M be a matroid and suppose that $\sigma = (e_1, e_2, \ldots, e_n)$ is a cyclic $(t-1, t)$-ordering of $E(M)$, where n is sufficiently large, and $t \geq 3$.

- *Then n is even,*
- *and there is a unique t-element circuit and a unique t-element cocircuit containing X_i.*

Furthermore, we can state precisely what these circuits and cocircuits are.

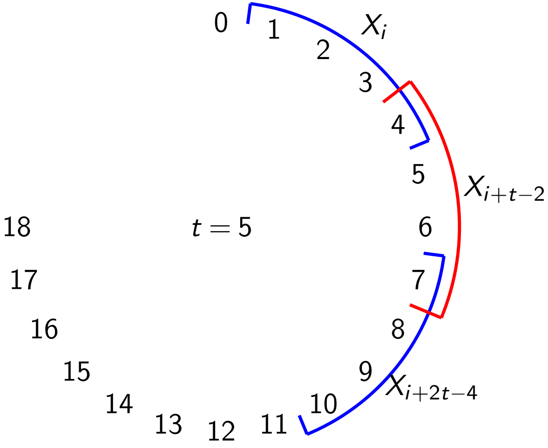
Helpful Tool

A circuit and a cocircuit of a matroid cannot intersect in exactly one element.



Lemma 1

Suppose that c_i and c_{i+2t-4} are far away.



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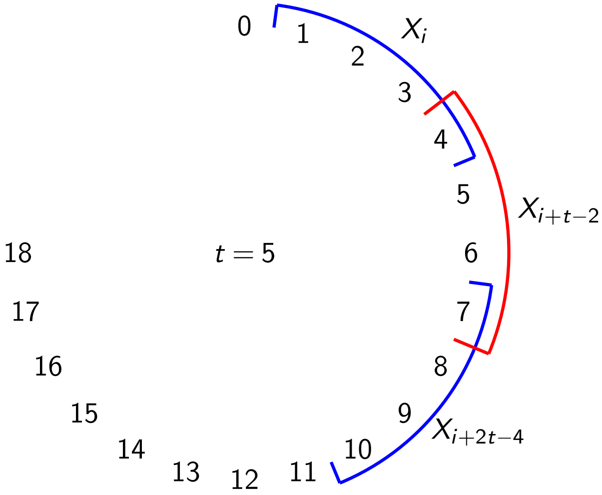

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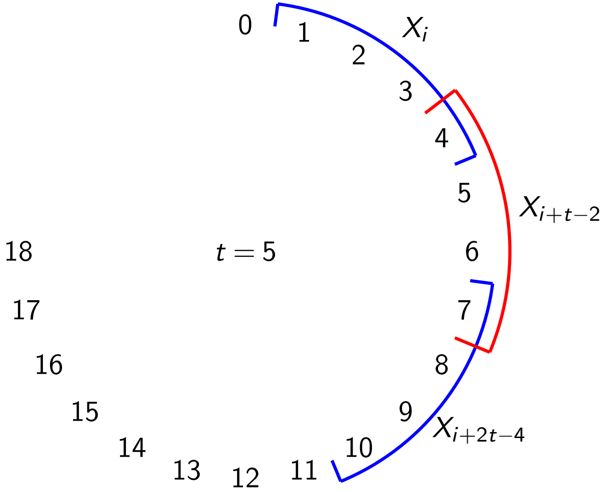

$|C_i \cap C^*_{i+t-2}| \geq 2$,
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$|C_{i+2t-4} \cap C^*_{i+t-2}| \geq 2$,
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so $c_i = c_{i+2t-4}$

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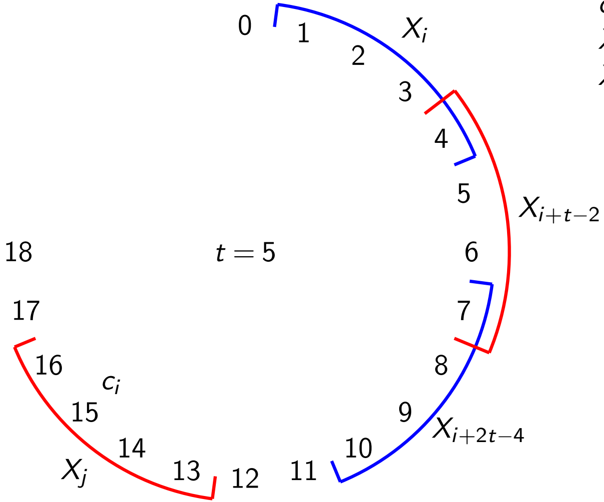


We want

$c_i \in X_j$

$X_j \cap X_i = \emptyset$

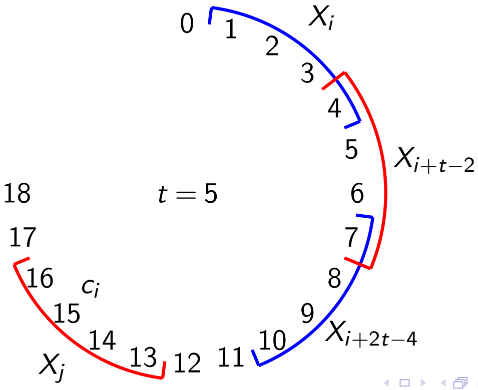
$X_j \cap X_{i+2t-4} = \emptyset$

Lemma 1

Lemma (1)

Let $n \geq 4t - 6$. *For all* $i \in [n]$,

1. *either* $C_i \subseteq \sigma_{[i,i+3t-6]}$ *or* $C_{i+2t-4} \subseteq \sigma_{[i,i+3t-6]}$, *and*
2. *either* $C_i^* \subseteq \sigma_{[i,i+3t-6]}$ *or* $C_{i+2t-4}^* \not\subseteq \sigma_{[i,i+3t-6]}$.



Lemma 2

Lemma (2)

Let $n \geq 4t-6$. *For all* $i \in [n]$,

$$C_i, C_i^* \subseteq \sigma_{[i-(2t-4),\, i+3(t-2)]}.$$

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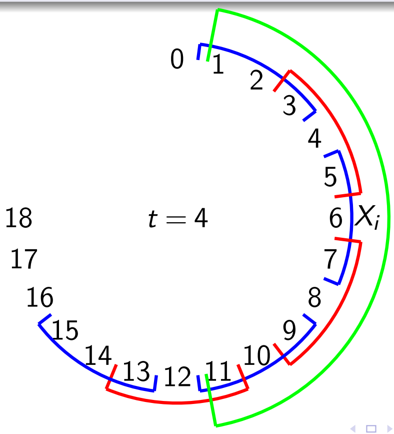

Proof of Main Theorem

Lemma (3)

If $n \geq 6t - 10$, *then*

$C_i \subseteq \sigma[i-1, i+t-1]$ *and* $C_i \subseteq \sigma[i-1, i+t-1]$.

Proof of Main Theorem

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If $n \geq 6t - 10$*, then*

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Corollary (4)

If $n \geq 6t - 10$*, then there is only one* t*-circuit containing* X_i *and only one* t*-cocircuit containing* X_i.

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If $n \geq 6t-10$, *then there is only one* t*-circuit containing* X_i *and only one* t*-cocircuit containing* X_i.

Corollary (5)

If $n \geq 6t-10$, $C_i = \sigma[i, i+t-1]$, *and* $j \equiv i \pmod 2$ *then*
$C_j = \sigma[j, j+t-1]$ *and* $C_{j+1} \subseteq \sigma[j, j+t-1]$.

Theorem [1]

Theorem

Suppose that $n \geq 6t - 10$ *and* $t \geq 3$. *Then* n *is even and, for all* $i \in [n]$, *there is a unique t-element circuit and a unique t-element cocircuit containing* $X_i\}$. *Moreover, up to parity,*

- *If* t *is odd, then*
  - $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ *is a t-element circuit, when* i *is odd, and a t-element cocircuit, when* i *is even.*
- *If* t *is even, then:*
  - $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ *is a t-element circuit circuit and cocircuit, when* i *is odd.*

Well behaved (t-1,t)-cyclic matroids

We say that M is *well behaved* if ($n \geq t-1$, and) there exists a cyclic ordering $\sigma = (e_1, e_2, \ldots, e_n)$ of $E(M)$ such that, for all odd $i \in \{1, 2, \ldots, n\}$, either

- $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is a t-element circuit and $\{e_{i+1}, e_{i+2}, \ldots, e_{i+t}\}$ is a t-element cocircuit, or
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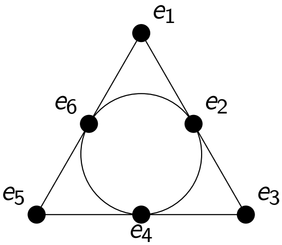

$\sigma_1 = (e_1, e_2, e_3, e_4, e_5, e_6)$ and $\sigma_2 = (e_4, e_2, e_6, e_1, e_3, e_5)$ are cyclic $(2,3)$-orderings.

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- $\{e_i, e_{i+1}, \ldots, e_{i+t-1}\}$ is a t-element circuit and t-element cocircuit.

Lemma (Lemma 6)

Let $t \geq 1$ and let M be a t-cyclic matroid. Then $|E(M)| \geq 2t-2$.

Size Lemma

Lemma (6)

Let $t \geq 1$ *and let* M *be a well behaved* $(t-1, t)$*-cyclic matroid. Then* $|E(M)| \geq 2t - 2$.

Construction

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- Let M' be the truncation of M.

Construction

- Let M be well behaved a $(t-1,t)$-cyclic matroid with $n \geq 2(t+2)-2$.
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- M' is obtained by freely adding an element, f, to M to get M_1 and then contracting f from M_1 to get M'.

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- Suppose $\{e_{j+1}, e_{j+2}, \ldots, e_{j+t}\}$ and $\{e_{j+3}, e_{j+4}, \ldots, e_{j+t+2}\}$ are t-element cocircuits of M, then
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- So $E(M) - \{e_{j+1}, e_{j+2}, \ldots, e_{j+t+2}\}$ is a hyperplane of M'.

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- $\{e_{j+1}, e_{j+2}, \ldots, e_{j+t+2}\}$ is a cocircuit.
- Let N be the Higgs lift of M'.

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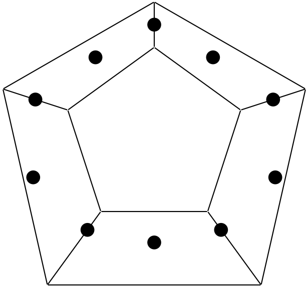
- By duality, we get the right $(t+2)$-circuits.



Conjecture

- Let M be well behaved $(t-1,t)$-cyclic matroid with $n \geq 2(t+2)-2$.
- M' is obtained by **not necessarily freely**, adding an element, f, to M to get M_1 and then contracting f from M_1 to get M'.
- Let M_1' be the matroid obtained by **not necessarily freely**, coextending M' by an element, g, and then deleting g.
- Then N has a well behaved-$(t+2)$-cyclic ordering.

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Conjecture

Let t be an integer exceeding two, and let M be a t-cyclic matroid.

- *If t is even, then M can be obtained from a spike or a swirl by a sequence of inflations.*
- *If t is odd, then M can be obtained from a wheel or whirl by a sequence of inflations.*

Thank You!