The Changing Face of Graph Saturation

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The Basis: Saturation

Definition

Given a graph H, a graph G of order n is said to be ***H*-saturated** provided G contains no copy of H, but the addition of any edge from the complement of G creates a copy of H. That is, given $e \in \bar{G}$, then $G + e$ contains a copy of H.

Extremal Numbers - the original question

Definition

The **maximum number** of edges in an H-saturated graph of order n is called the **extremal number (or the Turan number)** for H, and is denoted as $\mathrm{ex(n, H)}$.

Foundational Results on Extremal Numbers

Theorem

Turan, 1941

The unique graph with the maximum number of edges containing no a copy of K_p (for $p \geq 3$) is the complete balanced $(p-1)$-partite graph.

For triangles, this is the balanced complete bipartite graph, thus $\mathrm{ex}(n, K_3) = \lfloor n/2 \rfloor \lceil n/2 \rceil$. (Determined by W. Mantel et al. in 1906.)

Foundational Results on Extremal Numbers

Theorem

Erdős - Stone, 1946

$$\lim_{n\to\infty} \frac{\mathrm{ex}(\mathrm{n}, \mathrm{G})}{n^2} = \frac{1}{2}(1 - \frac{1}{\chi(G) - 1}).$$

Saturation Numbers - variation 1

Definition

The **minimum size** of an H-saturated graph of order n is called the **saturation number** and is denoted as $\mathrm{sat(n, H)}$.

Foundational Results on Saturation Numbers - Complete Graphs

In 1964 **Erdős, Hajnal and Moon** determined that:

Theorem

$$\mathrm{sat}(n, K_t) = (t-2)(n-1) - \binom{t-2}{2}.$$

Note: Zykov (1949) also introduced the idea, but in Russian so it remains mostly unknown.

The saturation graph for cliques

The graph $K_{t-2} \vee \overline{K}_{n-t+2}$, where $\vee$ denotes join.

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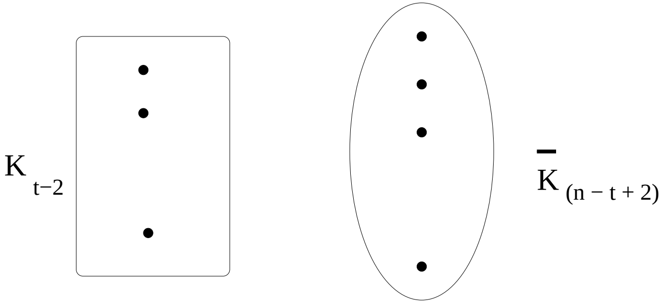
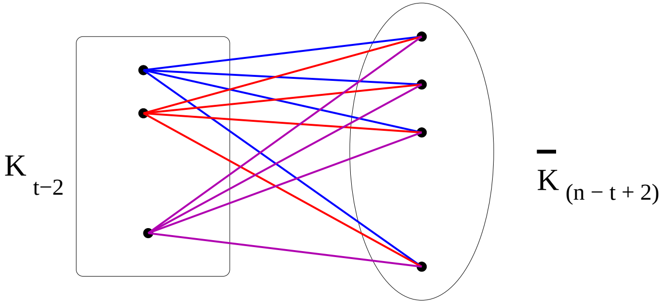


The graph $K_{t-2} \vee \overline{K}_{n-t+2}$, where $\vee$ denotes join.



Example: For a triangle, $\mathrm{sat}(n, K_3) = n - 1$.

Foundation Results - Kászonyi and Tuza, 1986

Theorem

For every graph F there exists a constant c such that

$$\mathrm{sat(n, F) < cn.}$$

NOTE: **All** saturation numbers are linear in n, while extremal numbers are **usually** quadratic in n.

Useful properties of extremal numbers

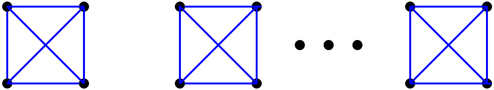
Monotone properties:

Let **F** be a family of graphs. Then $\mathrm{ex}(\mathrm{n}, \mathbf{F})$ satisfies:

1. $\mathrm{ex}(\mathrm{n}, \mathbf{F}) \leq \mathrm{ex}(\mathrm{n}+1, \mathbf{F})$.
2. If $\mathbf{F_1} \subset \mathbf{F}$ then $\mathrm{ex}(\mathrm{n}, \mathbf{F_1}) \leq \mathrm{ex}(\mathrm{n}, \mathbf{F})$.
3. If $H \subseteq G$, then $\mathrm{ex}(\mathrm{n}, \mathrm{H}) \leq \mathrm{ex}(\mathrm{n}, \mathrm{G})$.

Problems with saturation numbers

However, these rules do not hold in general for saturation numbers. Example of 3. Consider K_4 and a supergraph H obtained by attaching an additional edge to K_4. We know that $\mathrm{sat}(\mathrm{n}, \mathrm{K}_4) = 2\mathrm{n} - 3$. But for H we have:



$$here\ \ n = 4m\ \ and\ \ size = 6m$$

$$hence\ \ size = \ 3n/2$$

Thus, $\mathrm{sat}(\mathrm{n}, \mathrm{H}) \leq 3\mathrm{n}/2$.

Example Result

One of the early particular results on saturation numbers is due to Olleman, 1972.

Theorem

$$\mathrm{sat}(\mathrm{n}, \mathrm{C}_4) = \left\lfloor \frac{3\mathrm{n}-5}{2} \right\rfloor .$$

Definition

The set of all sizes of graphs on n vertices that are H-saturated is called the **saturation spectrum of H**.

In 1995 **Barefoot, Casey, Fisher, Fraughnaugh and Harary**, showed the following:

Theorem

For $n \geq 5$, there exists a K_3-saturated graph of order n with m edges if and only if it is complete bipartite or

$$2n - 5 \leq m \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

Since $\mathrm{sat(n, K_3) = n - 1}$, there is a gap at the bottom.

This gap is between $n - 1$ and $2n - 5$. This is a result of a combination of connectivity (the star is the unique graph with connectivity 1 that is K_3-saturated and the fact that triangle saturated graphs have diameter 2 (this forces more edges).
It is then easy to show the gap at the bottom exists. At the top, extremal theory and convexity suffice.

With **K. Amin, J. Faudree and E. Sidorowicz** (2013) we were able to generalize this result for all $t \geq 3$.

Theorem

For $n \geq 3t + 4$ and $t \geq 3$, there is a K_t-saturated graph G of order n with m edges if, and only if, G is complete $(t-1)$-partite or

$$(t-1)(n-t/2) - 2 \leq m \leq \left\lfloor \frac{(t-2)n^2 - 2n + (t-2)}{2(t-1)} \right\rfloor + 1.$$

Note, same sort of gaps exist. Also note this reduces to the Barefoot et al. result when $t = 3$.

Saturation number for paths

Kászonyi and Tuza, 1986:

Theorem

1. *For* $n \geq 3$, $\mathrm{sat}(n, P_3) = \lfloor n/2 \rfloor$.
2. *For* $n \geq 4$,
$$\mathrm{sat}(n, P_4) = \begin{cases} n/2 & n \text{ even} \\ (n+3)/2 & n \text{ odd.} \end{cases}$$
3. *For* $n \geq 5$, $\mathrm{sat}(n, P_5) = \left\lceil \frac{5n-4}{6} \right\rceil$.
4. *Let*
$$a_k = \begin{cases} 3 \cdot 2^{t-1} - 2 & \text{if} \quad k = 2t \\ 4 \cdot 2^{t-1} - 2 & \text{if} \quad k = 2t+1. \end{cases}$$

then if $n \geq a_k$ *and* $k \geq 6$, $\mathrm{sat}(n, P_k) = n - \left\lfloor \frac{n}{a_k} \right\rfloor$.

ex(n, P_k) - by several including Faudree and Schelp

Theorem

For all $n \geq 3$,

1.

$$\mathrm{ex}(\mathrm{n}, \mathrm{P}_4) = \begin{cases} n \text{ if } n \equiv 0 (mod 3) \\ n - 1 \text{ if } n \equiv 1, 2 (mod 3). \end{cases}$$

2.

$$\mathrm{ex}(\mathrm{n}, \mathrm{P}_5) = \begin{cases} 3n/2 \text{ if } n \equiv 0 (mod 4) \\ 3n/2 - 2, \text{ if } n \equiv 2 (mod 4) \\ 3(n-1)/2, \text{ if } n \equiv 1, 3 (mod 4). \end{cases}$$

3.

$$\mathrm{ex}(\mathrm{n}, \mathrm{P}_6) = \begin{cases} 2n, \text{ if } n \equiv 1 (mod 5) \\ 2n - 2, \text{ if } n \equiv 1, 4 (mod 5) \\ 2n - 3, \text{ if } n \equiv 2, 3 (mod 5). \end{cases}$$

Small Path Sat Spectrum

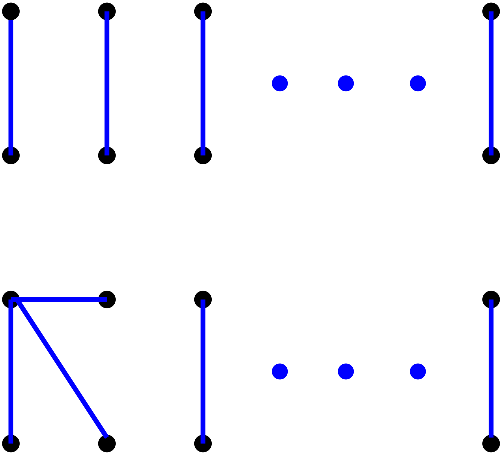
Work with **W. Tang, E. Wei, C.Q. Zhang** (2012).
If we consider P_3 it is simple to see that:

Theorem

$$\mathrm{sat}(n, P_3) = \mathrm{ex}(n, P_3) = \lfloor n/2 \rfloor.$$

There is a simple procedure for evolving a P_4-saturated graph from the saturation number to the extremal number, one edge at a time. Thus, the spectrum for P_4 is complete.

P_4 has a continuous saturation spectrum



Do any other graphs have a interval spectrum?

With J. Faudree, R. Faudree, M. Jacobson and B. Thomas (2009):

Theorem

If $t \geq 3$ and $n \geq t + 1$, then the saturation spectrum of the star $K_{1,t}$ is an interval from $\mathrm{sat(n, K_{1,t})}$ to $\mathrm{ex(n, K_{1,t})}$.

For completeness

Kászonyi and Tuza:

Theorem

$$\mathrm{sat}(n, K_{1,t}) = \begin{cases} \binom{t}{2} + \binom{n-t}{2} & \text{if } t+1 \leq n \leq t + t/2 \\ \lceil \frac{t-1}{2} n \rceil - t^2/8 & \text{if } t + 1/2 \leq n. \end{cases}$$

Folklore??? Obvious!

Theorem

$\mathrm{ex}(n, K_{1,t}) = \lfloor \frac{t-1}{2} n \rfloor$. *That is, a graph that is $t-1$-regular or nearly regular.*

P_5 with Tang, Wei and Zhang

Here things get a little bit more complicated.

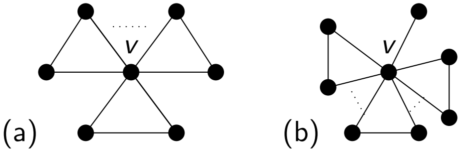
Theorem

Let $n \geq 5$ and $\mathrm{sat(n, P_5) \leq m \leq ex(n, P_5)}$ be integers. Then there exists an (n, m) P_5-saturated graph if and only if $n \equiv 1, 2(mod4)$, or

$$m \neq \begin{cases} \frac{3n-5}{2} \text{ if } n \equiv 3(mod4) \\ \frac{3n}{2} - j, j = 1, 2, \text{ or } 3 \text{ if } n \equiv 0(mod4). \end{cases}$$

The extremal number for $K_4 - e$ is achieved by the complete bipartite graph.

$\mathrm{sat}(n, K_4 - e) = \lfloor \frac{3(n-1)}{2} \rfloor$, and is achieved by:



Saturation spectrum for $K_4 - e$

With Jessica Fuller we showed:

Theorem

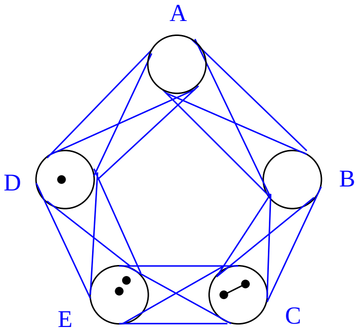
If G is a $K_4 - e$ saturated graph on n vertices, then either G is a complete bipartite graph, a 3-partite graph (like the saturation graph of the previous frame), or has size in the interval

$$[2n-4, \quad \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - n + 6]$$

Here the gap between the saturation number and 2n-4 happens for reasons similar to that for triangles.

A look at the proof

Case: Suppose $4n - 18 \leq m \leq \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - n + 5$.
Here $|A| = n - |B| - |C| - 5$, $|B| = b \geq 2$, $|C| = c \geq 2$, $|D| = 2$ and $|E| = 3$.



Then $m = (n-c)(c+2) - 5c + b - 4$. So as b increases by 1, with c fixed, then m increases by 1.

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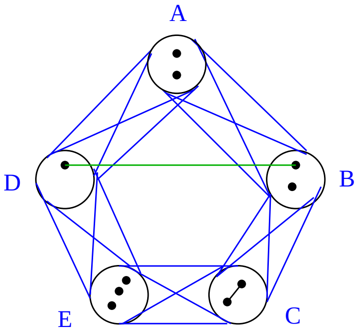

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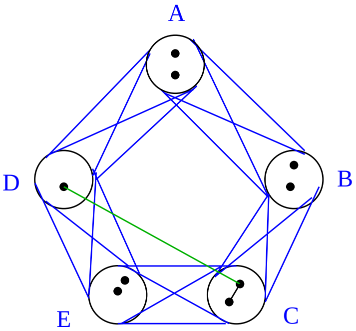


Then $m = (n - c)(c + 2) - 5c + b - 4$. So as b increases by 1, with c fixed, then m increases by 1.

Larger Cliques Minus an Edge

It is straight-forward to extend the $K_4 - e$ interval to larger cliques:

Theorem

There are $K_t - e$ saturated graphs in the interval
$$\left[(t-2)n - \binom{t-1}{2} - 1, \quad \left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil + (t-3)n - \binom{t-2}{2} - 1\right].$$

Also, there are $(K_t - e)$-saturated graphs for sporadic values of m between

$$\left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil + (t-3)n - \binom{t-2}{2} + 4 \text{ and}$$
$$\left\lfloor \frac{n-t}{2} \right\rfloor \left\lceil \frac{n-t}{2} \right\rceil + (t-2)n - \binom{t-1}{2} - 1.$$

Extremal Number for the Fan F_k

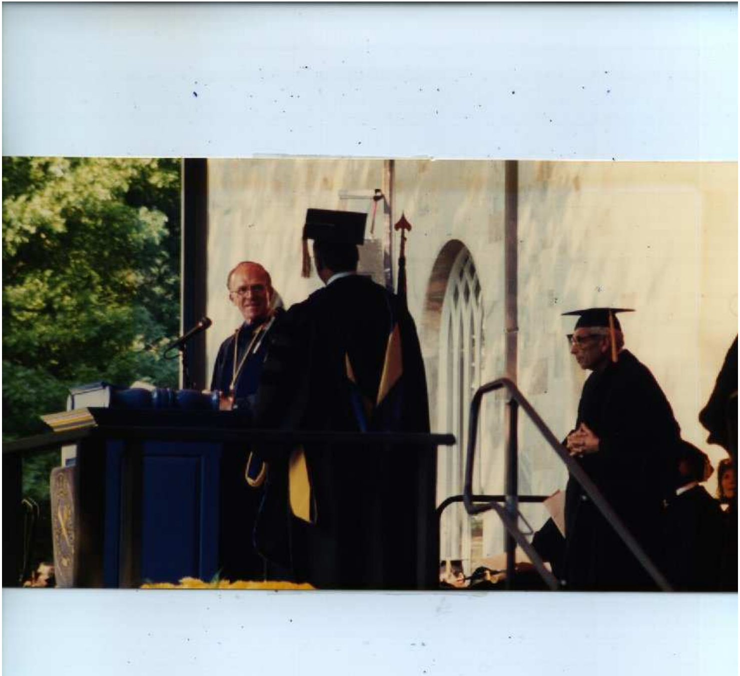
With Erdős, Furedi and Gunderson (1995) we determined the extremal number for fans F_t.

Theorem

For every $t \geq 1$, and for every $n \geq 50t^2$, if a graph G on n vertices has more than

$$\left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} t^2 - t & \text{if } t \text{ is odd} \\ t^2 - \frac{3}{2}t & \text{if } t \text{ is even} \end{cases}$$

edges, then G contains a copy of the t-fan, F_t. Furthermore, the number of edges is best possible.



Saturation numbers for fans

With J. Fuller we showed the following:

Theorem

For $t \geq 2$, and $n \geq 3t - 1$, $\mathrm{sat(n, F_t) = n + 3t - 4.}$

Theorem

There exists an F_2-saturated graph G on $n \geq 7$ vertices and m edges where $m = n + 2$, or $2n - 4 \leq m \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 2$, or m is the size of a complete bipartite graph with one additional edge.

Another Variation - due to Bollobás, 1967

Definition

A graph F is weakly G-saturated if F does not contain a copy of G, but there is an ordering of the missing edges of G so that if they are added one at a time, each edge creates a new copy of F. The minimum size of a weakly F-saturated graph G of order n is denoted $\mathrm{wsat(n, F)}$.

Note:

$$\mathrm{wsat}(n, H) \leq \mathrm{sat}(n, H),$$

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since any ordering works for an *H*-saturated graph.

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$$\mathrm{wsat(n, H)} \leq \mathrm{sat(n, H)},$$

since any ordering works for an *H*-saturated graph.

Interesting when we can find such an ordering on a graph that is not *H*-saturated.

Foundational Result on Weak Saturation

The following result was conjectured by Bollobás for all k and verified for $3 \leq k \leq 7$.

Theorem

Lovász, 1977 and new proof by Kalai, 1984

For integers n and k,

$$\mathrm{wsat}(\mathrm{n}, \mathrm{K_k}) = \mathrm{sat}(\mathrm{n}, \mathrm{K_k}) = \binom{\mathrm{k}-2}{2} + (\mathrm{k}-2)(\mathrm{n}-\mathrm{k}+2).$$

More Key Results on Weak Saturation

Borowiecki and Sidorowicz (2002) - Cycles:

Theorem

(1) *For* $n \geq 2k+1$, $\mathrm{wsat(n, C_{2k+1}) = n - 1.}$

(2) *For* $n \geq 2k$, $\mathrm{wsat(n, C_{2k}) = n.}$

Recall: Ollman showed $\mathrm{sat}(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor$.

But $\mathrm{wsat}(n, C_4) = n$. Hence,
we know that $\mathrm{wsat}(n, H)$ is not equal to $\mathrm{sat}(n, H)$ in general.

Question:

Question

For which graphs G is $\mathrm{sat(n, G)} = \mathrm{wsat(n, G)}$*?*

Bipartie Saturation: Introduced by Erdős, Hajnal, and Moon. We seek the minimum number of edges in an H-free bipartite graph with n vertices in each partite set. This definition is only meaningful if H is bipartite.

Conjecture

$\mathrm{sat}(\mathrm{K}_{\mathrm{n,n}}, \mathrm{K}_{\mathrm{s,t}}) = \mathrm{n}^2 - (\mathrm{n} - \mathrm{s} + 1)^2.$

Gan, Korándi, and Sudakov, (2015)

Theorem

Let $1 \leq s \leq t$ be fixed integers and $n \geq t$. Then

$$\mathrm{sat}(K_{n,n}, K_{s,t}) \geq (s+t-2)n - (s+t-2)^2.$$

Variation - Ordering

In the bipartite setting add the additional restriction:

Order the two partite sets of H and of G, then require that each missing edge create a copy of H respecting these orderings. This means that the first class of H lies in the first class of G.

Wessel (1966) and Bollobas (1967) independently showed that the ordered saturation number of $K_{s,t}$ is $n^2 - (n - s + 1)(n + t + 1)$.

Extension - Multipartite Graphs: Sullivan and Wenger 2017

Theorem

Let ℓ be a positive integer. If n_i, for $i = 1, 2, 3$ are positive integers such that $n_1 \geq n_2 \geq n_3 \geq 32\ell^3 + 40\ell^2 + 11\ell$, then

$$\mathrm{sat}(\mathrm{K}_{\mathrm{n}_1,\mathrm{n}_2,\mathrm{n}_3}, \mathrm{K}_{\ell,\ell,\ell}) = 2\ell(\mathrm{n}_1 + \mathrm{n}_2 + \mathrm{n}_3) - 3\ell^2 - 3.$$

Theorem

Let ℓ be a positive integer. If n_i, $i = 1, 2, 3$ are positive integers such that $n_1 \geq n_2 \geq n_3 \geq 32(\ell - 1)^3 + 40(\ell n - 1)^2 + 11(\ell - 1)$, then

$$\mathrm{sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell-1}) = 2(\ell - 1)(n_1 + n_2 + n_3) - 3(\ell - 1)^2.$$

The t-colored **rainbow saturation number** $\mathrm{rsat_t(n, F)}$ is the minimum size of a t-edge-colored graph on n vertices that contains no rainbow colored copy of F (all edges colored differently), but the addition of any missing edge in any color creates a rainbow copy of F.

Let $\mathcal{R}(K_s)$ be the set of all rainbow colored copies of K_s.

Theorem

For constants c_1 and c_2,

$$c_1 \frac{n \ \log \ n}{\log \ \log \ n} \leq \mathrm{rsat_t(n, \mathcal{R}(K_s) \leq c_2 n} \ \log \ \mathrm{n.}$$

They further showed that the upper bound was of the right order of magnitude.

This was also shown by Korándi (2018) in a strong sense.

Theorem

For $s \geq 3$ and $t \geq \binom{s}{2}$, we have

$$\mathrm{rsat_t(n, K_s)} \geq \frac{\mathrm{t(1+o(1))}}{\mathrm{(t-s+2)}\ \log\ \mathrm{(t-s+2)}}\mathrm{n}\ \log\ \mathrm{n},$$

with equality for $s = 3$.

Rainbow stars

Theorem

Barrus et al. (2017) 1. If $t \geq k$ *and* $n \geq (k+1)(k-1)/t$ *then*

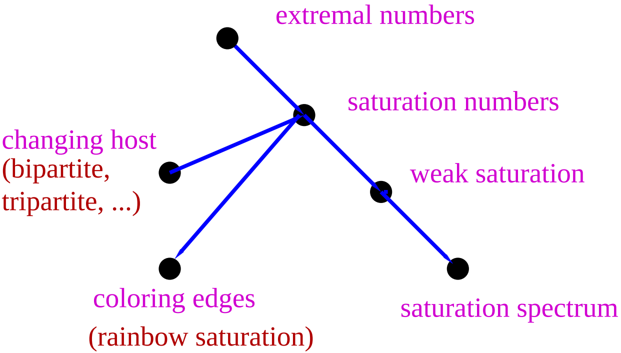
$$\mathrm{rsat_t}(\mathrm{n}, \mathcal{R}(\mathrm{K_{1,k}})) = (1 + \mathrm{o}(1))\frac{\mathrm{k}-1}{2\mathrm{t}}\mathrm{n}^2.$$

2. For all $k \geq 4$, $\mathrm{rsat_t}(\mathrm{n}, \mathcal{R}(\mathrm{P_k})) \geq \mathrm{n} - 1$.

Question

Is there a graph $G \neq K_{1,m}$ *such that* $\mathrm{rsat_t}(\mathrm{n}, \mathcal{R}(\mathrm{G})) = \theta(\mathrm{n}^2)$*?*

The Variations shown



Deeper Results: Saturation for unions of cliques

With R. Faudree, M. Ferrara, and M. Jacobson (2009):
First tK_p.

Theorem

Let $t \geq 1$, $p \geq 3$ and $n \geq p(p+1)t - p^2 + 2p - 6$ be integers. Then

$$\mathrm{sat}(n, tK_p) = (t-1)\binom{p+1}{2} + \binom{p-2}{2} + (p-2)(n-p+2).$$

$K_p \cup K_q$

Theorem

Let $2 \leq p \leq q$ and $n \geq q(q+1) + 3(p-2)$ be integers. Then

$$\mathrm{sat}(n, K_p \cup K_q) = (p-2)(n-p+2) + \binom{p-2}{2} + \binom{q+1}{2}.$$

Generalized Fans

Definition

Let the graph comprised of t copies of K_p intersecting in a common K_ℓ be called a generalized fan and be denoted $F_{p,\ell}$

Theorem

Let $p \geq 3$, $t \geq 2$ and $p-2 \geq \ell \geq 1$ be integers. Then, for sufficiently large n,

$$\mathrm{sat}(n, F_{p,\ell}) = (p-2)(n-p+2) + \binom{p-2}{2} + (t-1)\binom{p-\ell+1}{2}.$$

with G. Chen, F. Pfender and B. Wei 2003

Definition

A graph on $(r-1)k+1$ vertices consisting of k cliques each with r vertices, which intersect in exactly one common vertex, is called a $(k,r)-fan$.

Theorem

For every $k \geq 1$, and for every $n \geq 16k^3r^8$, if a graph G on n vertices has more than

$$\mathrm{ex(n, K_r)} + \begin{cases} k^2 - k & \text{if } k \text{ is odd} \\ k^2 - \frac{3}{2}k & \text{if } t \text{ is even} \end{cases}$$

edges, then G contains a copy of the (k,r)-fan. Furthermore, the number of edges is best possible.

To see the last result is best possible consider:

For odd k take the Turan graph and embed two vertex disjoint copies of K_k in one partite set.

For even k take the Turan graph and embed a graph with $2k-1$ vertices and $k^2-(3/2)k$ edges with max degree $k-1$ in one partite set.

Definition

A tree T of order ℓ, $T \neq K_{1,\ell-1}$, having a vertex that is adjacent to at least $\lfloor \frac{\ell}{2} \rfloor$ leaves is called a scrub-grass tree.

Theorem

Let T be a path or scrub-grass tree on $\ell \geq 6$ vertices and $n = |G| \equiv 0 \ mod(\ell - 1)$ and m be an integer such that $1 \leq m \leq \lfloor \frac{\ell-2}{2} \rfloor - 1$. There is no graph of size $\frac{n}{\ell-1}\binom{l-1}{2} - m$ in the spectrum of T. Hence, there is a gap in the spectrum.

If F is a graph of order p and size q:

Theorem

$$\frac{\delta n}{2} - \frac{n}{\delta + 1} \leq \mathrm{wsat(n, F)} \leq (\delta - 1)\mathrm{n} + (\mathrm{p} - 1)\frac{\mathrm{p} - 2\delta}{2}.$$

We further showed that for any tree T_p on p vertices:

Theorem

$$p-2 \leq \mathrm{wsat}(\mathrm{n}, \mathrm{T_p}) \leq \binom{\mathrm{p}-1}{2}.$$

weak sat for multiple copies

with R. Faudree (2014):

Theorem

$$\mathrm{wsat}(n, kK_t) = (t-2)n + k - (t^2 - 3t + 4)/2.$$

Theorem

$$\mathrm{wsat}(\mathrm{n}, \mathrm{kC_t}) = \begin{cases} n + k - 2 & \text{if } t \text{ is odd} \\ n + k - 1 & \text{if } t \text{ is even.} \end{cases}$$