Topological Properties of Maximal Linklessly Embeddable Graphs

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- K_n = the complete graph with n vertices.
- cG = the *complement of* G in K_n.
- $V(cG) = V(G)$, $E(cG) = \{\{i,j\} \mid \{i,j\} \notin E(G)\}$.

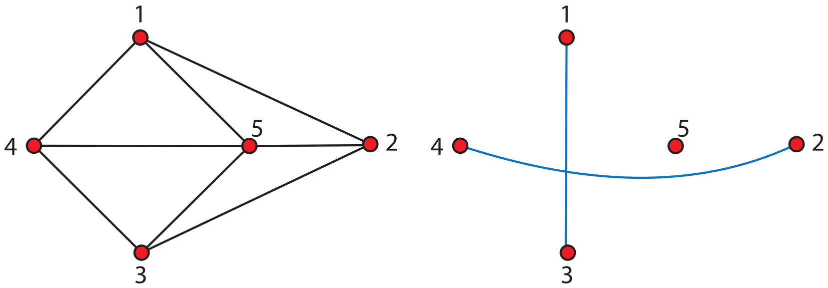
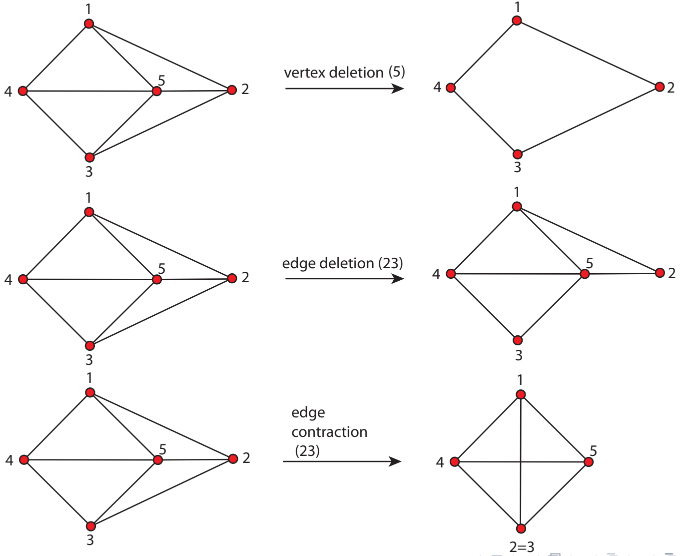
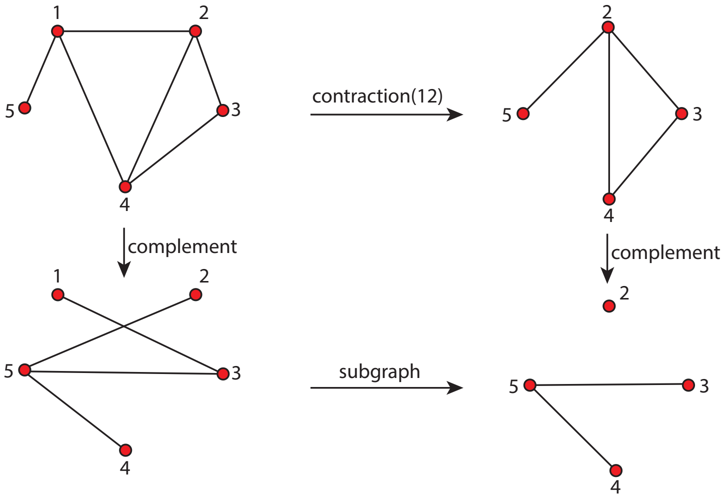


Figure: Complementary graphs

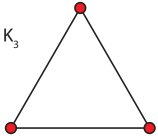
- For a graph G, *a minor of* G is any graph that can be obtained from G by a sequence of vertex deletions, edge deletions, and simple edge contractions.

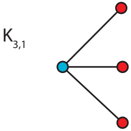


Contracting edges produces subgraphs in the complement.

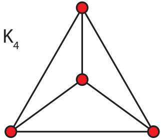


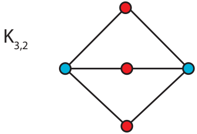
- (N. Robertson and P. Seymour, '83-'04) Every class of graphs closed under taking minors can be defined by a finite set of forbidden minors.
- A graph is a linear forest (disjoint union of paths) if and only if it does not have either of K_3 or $K_{1,3}$ as a minor.



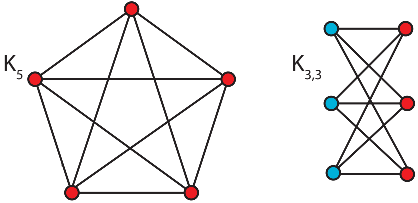


- A graph is outerplanar if and only if it does not have either of K_4 or $K_{2,3}$ as a minor.

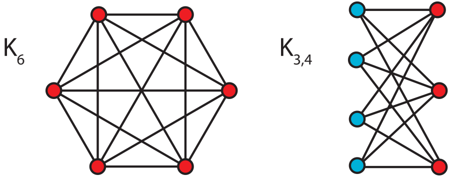




- (K. Wagner, 1937) A graph is planar if and only if it does not have either of K_5 or $K_{3,3}$ as a minor.



- (V. Sivaraman, 2017) A graph does not have either of K_6 or $K_{4,3}$ as a minor if and only if ...?(R. Nikkuni, Y. Tsutsumi 2012?)



- (J. Battle, F. Harary, Y. Kodama 1962) Every planar graph with nine points has a non planar complement.

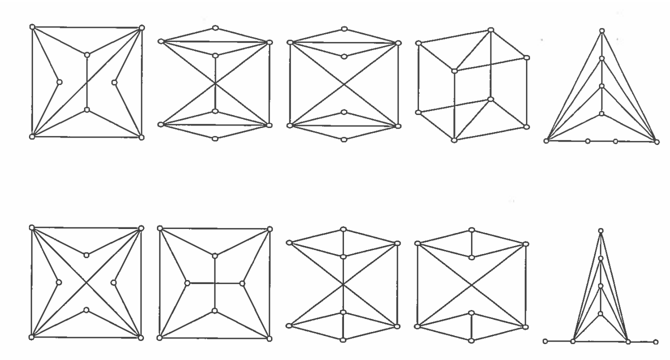


Figure: Self-complementary graphs on 8 vertices.

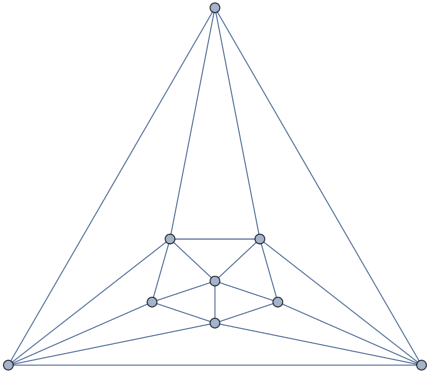
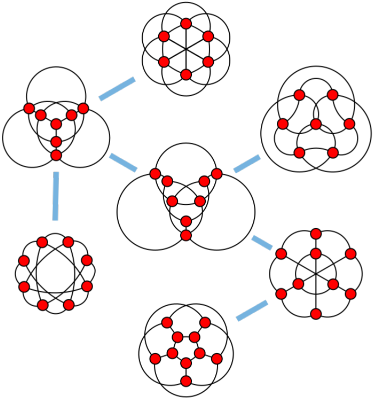


Figure: The Fritsch Graph.

- Every maximal planar graph of order n has exactly $3n - 6$ edges.

- (Tutte, 1960) Every maximal planar graph is 3-connected.
- (Fařy, 1948) Every planar 3-connected graph has a straight-edge planar embedding.
- A graph is called *intrinsically linked* (**IL**) if every one of its embeddings into $\mathbb{R}^3$ contains a nontrivial link. A graph that is not intrinsically linked is called *linklessly embeddable* (**nIL**).
- (Sachs 1983) Does every *nIL* graph admit a straight-edge linkless embedding into $\mathbb{R}^3$?

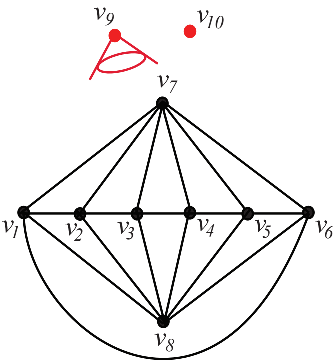
- (N. Robertson, P. Seymour, R. Thomas, 1993) A graph is nIL if and only if it does not have any of the Petersen family of graphs as a minor.



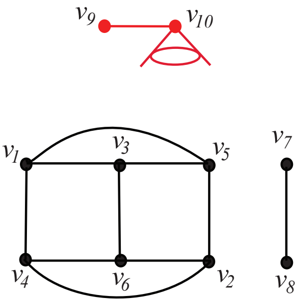
- What is the minimal value of n such that any graph nIL graph of order n has an IL complement?

- (Mader 1968) Any graph on n vertices and at least $4n - 9$ edges contains a K_6 minor.

- $n(n-1)/2 \geq 2(4n-9) \Rightarrow n^2 - 17n + 36 \geq 0 \Rightarrow n \geq 15.$



(a)



(b)


16
?
10

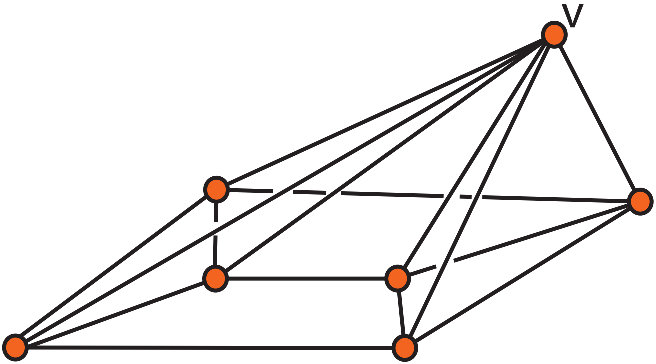
Y.C. de Verdière, 1987

Let $\mathbb{R}^{(n)}$ denote the space of real symmetric $n \times n$ matrices. If $G = (V, E)$ is a graph of order n, then $\mu(G)$ is the largest corank of any matrix $M = (M_{i,j}) \in \mathbb{R}^{(n)}$ such that:

- for all i, j with $i \neq j$, $M_{i,j} < 0$ if i and j are adjacent, and $M_{i,j} = 0$ if i and j are not adjacent;
- M has exactly one negative eigenvalue, of multiplicity 1;
- There is no nonzero matrix $X \in \mathbb{R}^{(n)}$ such that $MX = 0$ and such that $X_{i,j} = 0$ whenever $i = j$ or $M_{i,j} \neq 0$.

- If H is a minor of G, then $\mu(H) \leq \mu(G)$;
- $\mu(G) \leq 1$ if and only if G is a disjoint union of paths;
- $\mu(G) \leq 2$ if and only if G is outer planar;
- $\mu(G) \leq 3$ if and only if G is planar;
- $\mu(G) \leq 4$ if and only if G is nIL;
- $\mu(G) \leq 5$ if and only if G is ?;

- $\mu(G) \leq \mu(G - v) + 1, \forall v \in V(G)$;



- If G is a disjoint union of paths, then $\mu(cG) \geq n - 3$;
- If G is outer planar, then $\mu(cG) \geq n - 4$;
- If G is planar, then $\mu(cG) \geq n - 5$;
- If G is nIL, then $\mu(cG) \geq \ldots$?
- (A. Kotlov, L. Lovász, S. Vempala, 1996)
  $\mu(G) + \mu(cG) \geq n - 2$.

Theorem (A. Pavelescu, E. Pavelescu, 2019)

Let G denote a simple graph with 13 vertices. Then either G or cG is intrinsically linked.

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Conjecture

Let G denote a simple graph with 11 vertices. Then either G or cG is intrinsically linked.

Definition

A simple graph is called *triangular* if every edge is part of a triangle.

Theorem (R. Naimi, A. Pavelescu, E. Pavelescu, 2019)

The following statements are equivalent:

- *Every maximal linklessly embeddable (maxnil) graph has minimum degree 3.*
- *Every maximal linklessly embeddable graph is 3-connected.*
- *Every maximal linklessly embeddable graph is triangular.*

Let G be maxnil graph of order n. Then $4n - 10 \geq |E(G)| \geq ?$

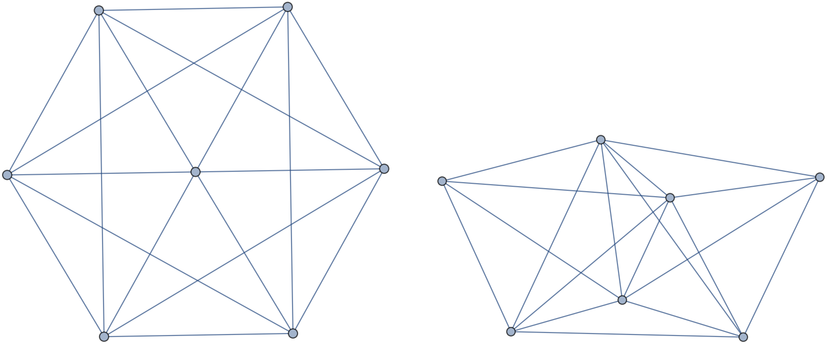


Figure: The Maxnils on 7 Vertices.

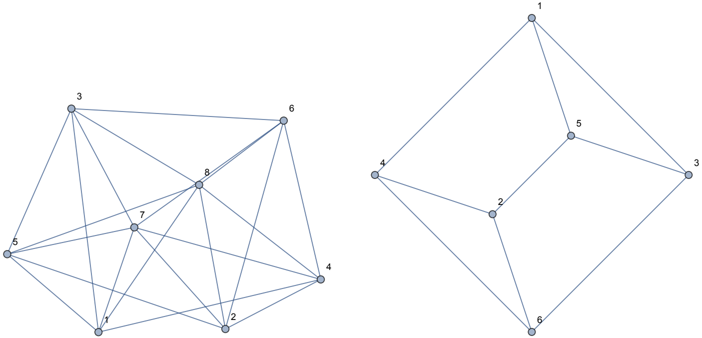


Figure: Jørgensen's Graph

Jørgensen's graph is maxnil, has order 8, and has $4 \times 8 - 11 = 21$ edges.

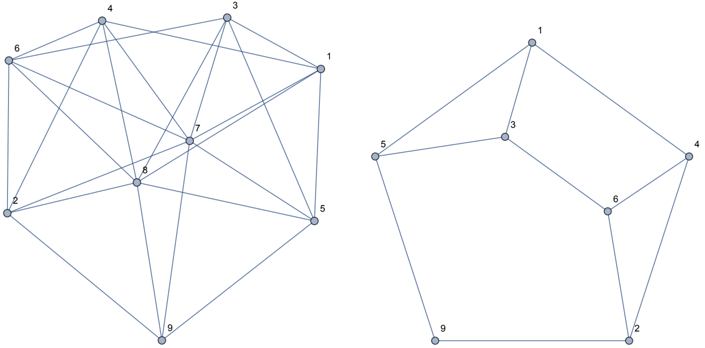
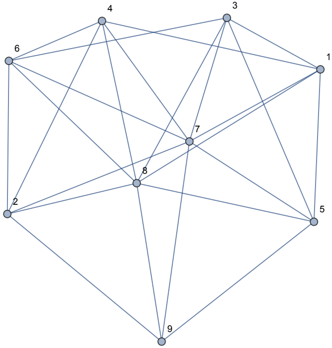


Figure: Jørgensen's Graph + subdivision.

The new graph has order 9 and 24 edges.



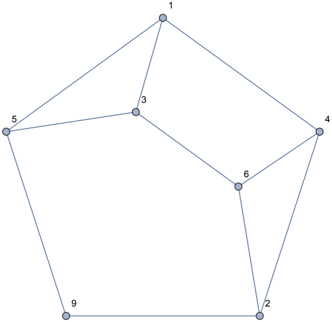
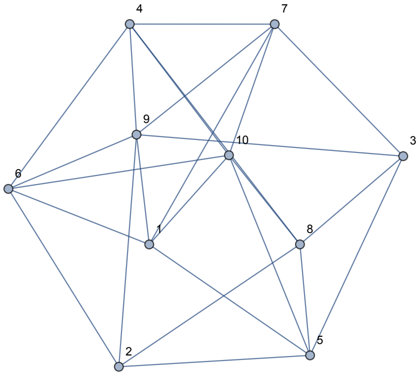


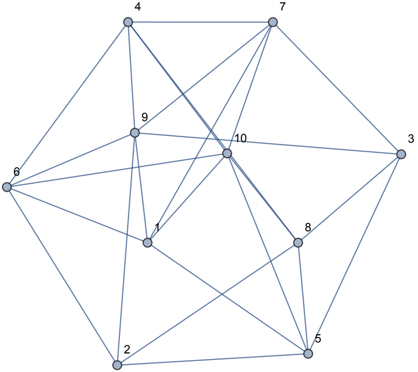
Figure: Jørgensen's Graph + subdivision.

The new graph has order 9 and 24 edges.

Notice that $24 = 3 \times 9 - 3$. Is the lower bound $3n - 3$?



This graph is maxnil, it has order 10, and it has 25 edges.



This graph is maxnil, it has order 10, and it has 25 edges.

Conjecture

Let G be a linklessly embeddable graph of order n. Then $|E(G)| \geq 3n - 5$.

Abdee, abdee, abdee, that's all folks!