Switching, Local Complementation and Pointed Swaps in Binary matroids

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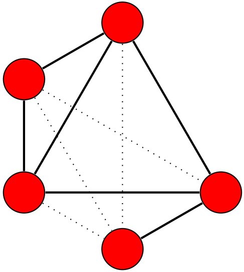
7th Annual Mississippi Discrete Math Workshop , October 2019

Three Graph Operations

- **Complementation**: Complement inside K_n



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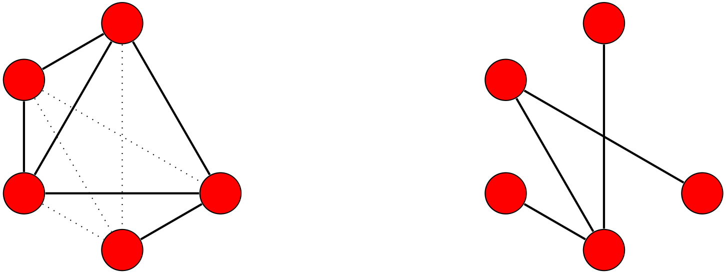

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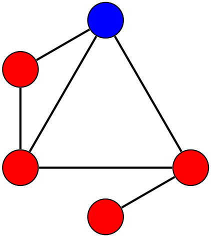

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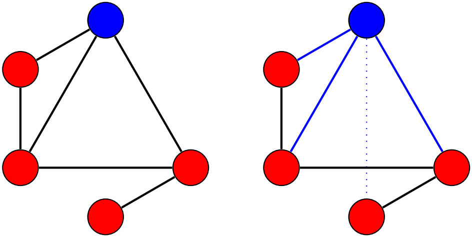

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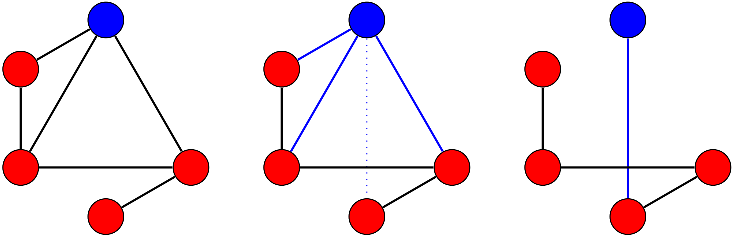

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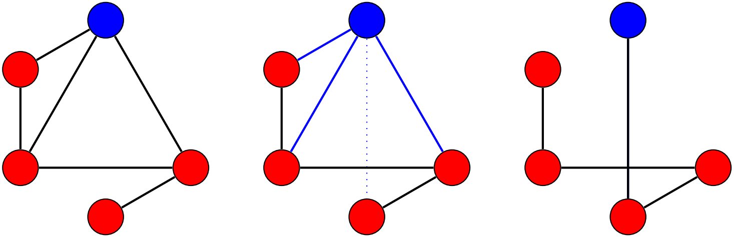

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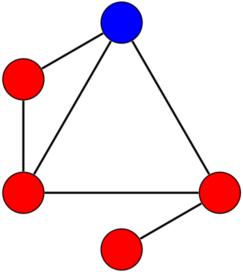

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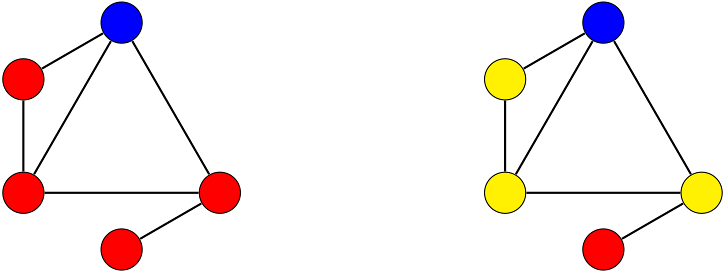

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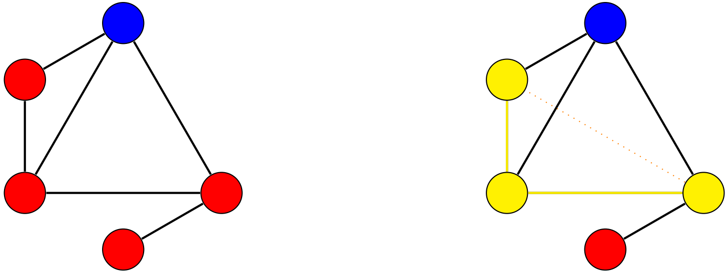

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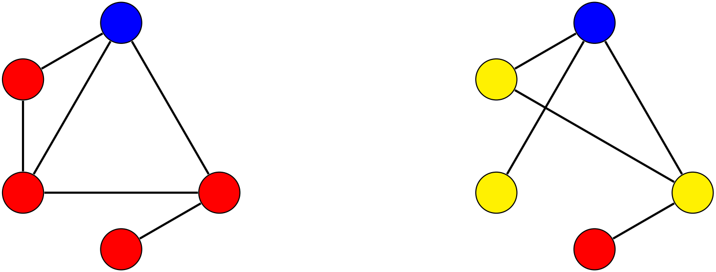

All Graphs Obtainable

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Theorem

All n-vertex graphs can be obtained from K_n via a sequence of switchings and local complementations.

Matroid prerequisites

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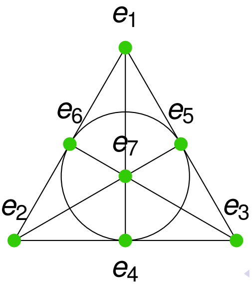

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- All rank r binary matroids : Restrictions of P_r.

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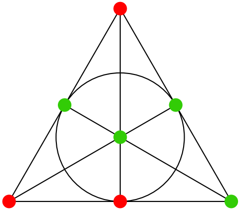
- Hyperplane : Closed set of rank $r - 1$.
- Also, complements of cocircuits.

Main Idea

- Want all binary matroids of rank at most r starting with P_r.
- Operations:
    - Complementation
    - Switching
    - Local complementation

Binary Matroid Analogues

- **Complementation** (inside fixed projective geometry P_r)



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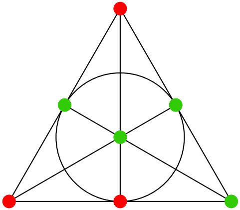

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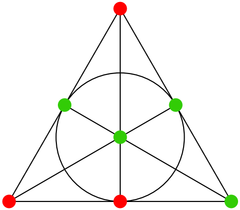

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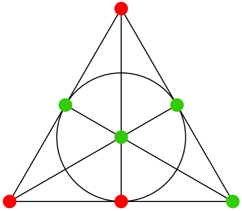

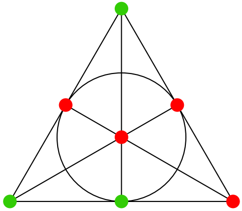

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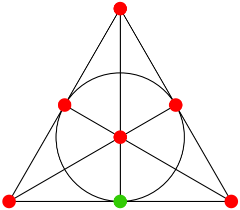

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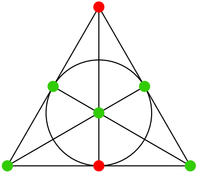

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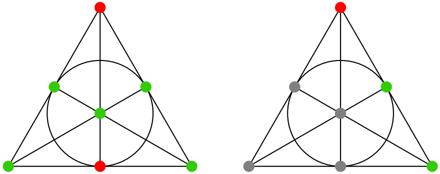

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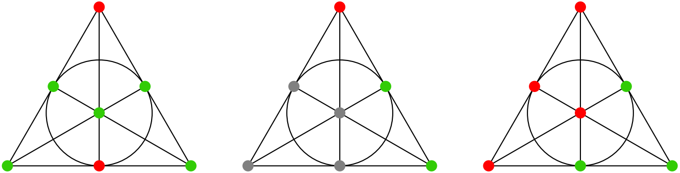

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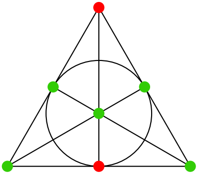

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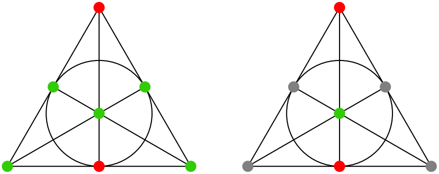

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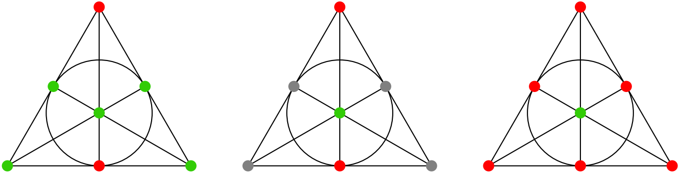

- $P(U_{2,3}, U_{2,3}) \xrightarrow{\text{switching}} U_{1,1}$

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Theorem

Matroids obtainable from P_r using switchings and complementation are isomorphic to one of P_r, $U_{0,0}$, P_{r-1} and A_r.

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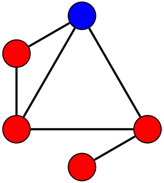

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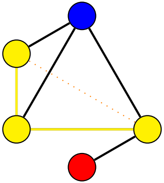

- Edges incident with blue vertex v : Complete vertex bond $C^* \cap G$.

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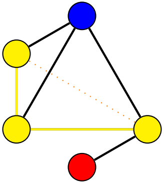

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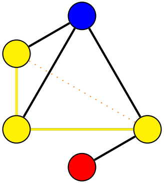

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- Binary Matroids : Complement inside $\mathrm{cl}_{P_r}(C^* \cap E(M)) - C^*$.

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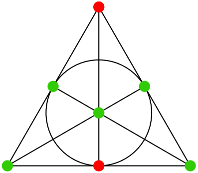

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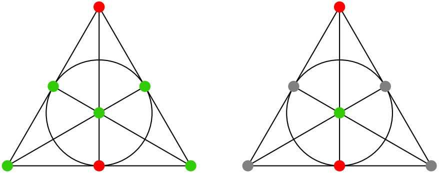

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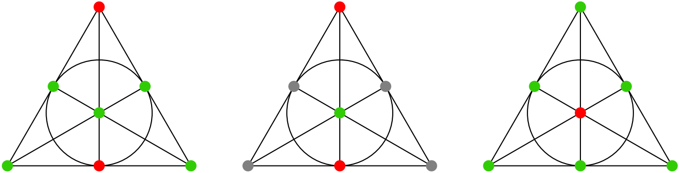

- $P(U_{2,3}, U_{2,3}) \xrightarrow{\text{L.C.}} M(K_4)$

Not all Binary Matroids are obtainable

Theorem (Oxley, Singh; 2019)

For $r > 4$, not all binary matroids of rank at most r can be obtained from P_r using complementation, switching, and local complementation.

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For $r > 4$, not all binary matroids of rank at most r can be obtained from P_r using complementation, switching, and local complementation.

- For $r \leq 4$, we can.

Coloring Notation

- Element e of P_r colored green : e is in $E(M)$.
- Colored red : Not in $E(M)$.

Proof

- **(Property 1)** : For every two distinct projective cocircuits C^* and D^*, red and green elements in $(C^* - D^*)$ both have rank $r - 1$.

Proof

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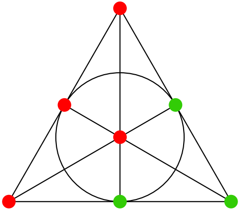
    - Composition of switching and complementation.
    - does not change Property 2.

Proof (continued)

- All colorings obtainable from X using given operations satisfy Property 2.

Pointed Swaps

- Off-Element Swaps



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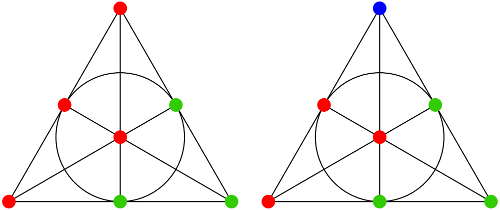

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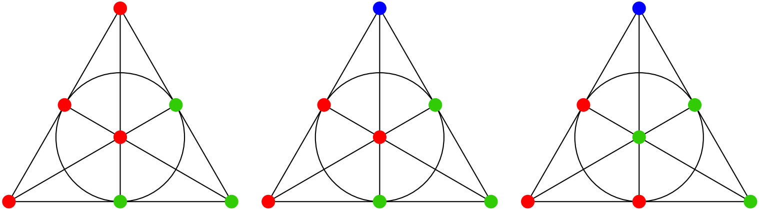

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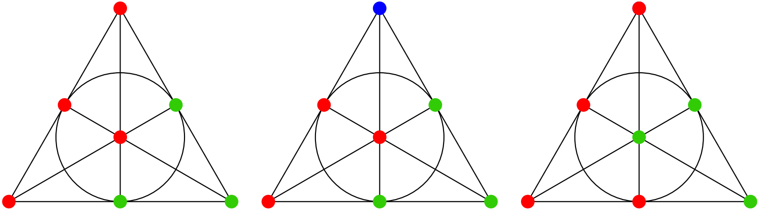

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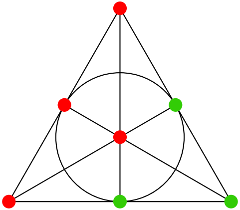

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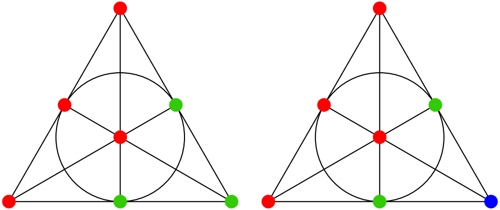

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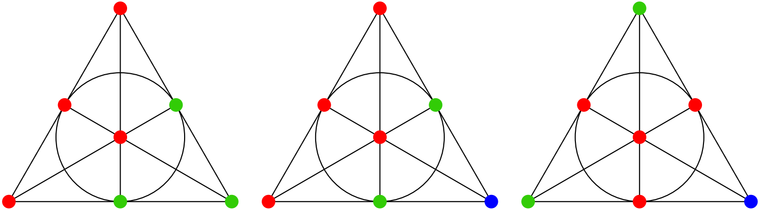

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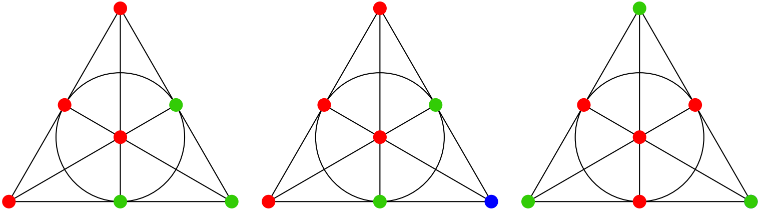

Pointed Swaps - matrix viewpoint

$$[v_1, \dots, v_k, \dots, v_r] \xrightarrow{\psi_{w}^{-}} [v_1 + w, \dots, v_k + w, \dots, v_r + w].$$

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$$[v_1, \ldots, v_k, \ldots, v_r, w] \xrightarrow{\psi_w^+} [v_1 + w, \ldots, v_k + w, \ldots, v_r + w, w].$$

- w : green element (On-swap).

Same element matroids obtainable via pointed swaps

Lemma

Let M be a t-element matroid that is a restriction of P_r. Then every t-element restriction of P_r can be obtained from M using pointed swaps.

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$$[v_1 + w, \ldots, v_r + w] \xrightarrow{\psi_{v_k+w}^+} [v_1 + v_k, \ldots, v_k + w, \ldots, v_r + v_k].$$

$$[v_1 + v_k, \ldots, v_k + w, \ldots, v_r + v_k] \xrightarrow{\psi_{v_k}^-} [v_1, \ldots, w, \ldots, v_r].$$

☐

All Matroids Obtainable

Theorem (Oxley, Singh; 2019)

For $r > 1$, all binary matroids of rank at most r can be obtained from P_r via :

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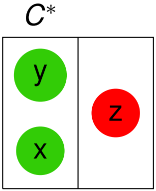
- $P_r \xrightarrow{Hyp.Comp.} A_r \xrightarrow{Ptd.Swaps} P_{r-1} \oplus U_{1,1} \xrightarrow{Hyp.Comp.} U_{1,1}$.
- Minimal counterexample M has ≥ 2 elements.

☐

Proof (continued)



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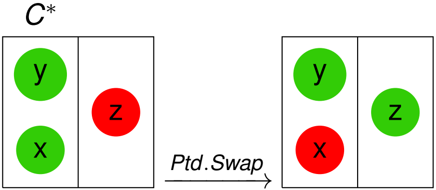

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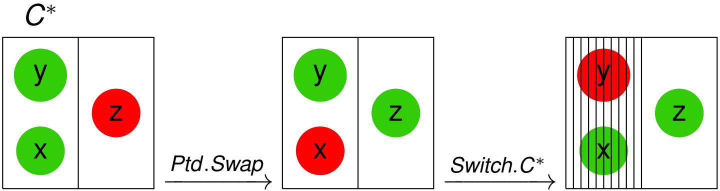

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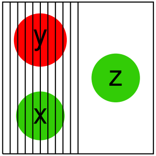

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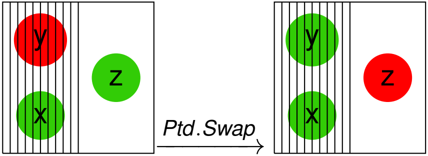

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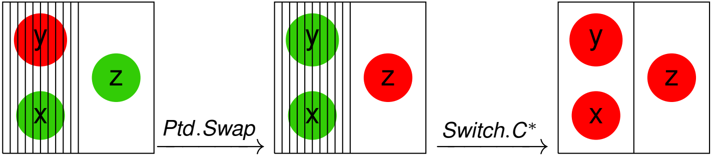

- Decreased the size of M.

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- Complementation inside hyperplanes and on-swaps are enough.

Local Complementation and Pointed Swaps

Theorem (Oxley, Singh; 2019)

All binary matroids of rank at most r with ≥ 2 elements can be obtained from P_r using local complementation and pointed swaps.

Proof (Sketch)

- If M has 2 coloops, then we can get M' with one more element using local complementation.

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- If M has 2 coloops, then we can get M' with one more element using local complementation.
- All matroids with size in $[2, 2^{r-2} + 2]$ are obtainable from $U_{2,2}$. Call them $\mathbb{M}_1$.
- $P_r \xrightarrow{L.C.} A_r$. B be a basis inside A_r. Pick $k-$ elements each of $A_r - B$ and $P_r - A_r$ and swap their colors.
- L.C. w.r.t $C^* = A_r$ gives a matroid with $(2^r - 1) - 2k$ elements. Note $k \in [0, 2^{r-1} - r]$.
- All matroids of odd size between $2^r - 1$ and $2r - 1$ are obtainable from P_r. Call them $\mathbb{M}_2$.
- $\mathbb{M}_1$ intersects $\mathbb{M}_2$.
- All matroids with odd size > 1 are obtainable from P_r.
- Similar argument for even size.

Thank You for your attention!