χ-boundedness: A tale of two graph invariants

Vaidy Sivaraman
Mississippi State University
(based on collaboration with Maria Chudnovsky, Kathie Cameron, Shenwei Huang, Irena Penev, Sang-il Oum, O-joung Kwon, Ringi Kim, Deven Gill)

Mississippi Discrete Math Workshop, Oxford, MS

October 26, 2019

The two invariants

- Chromatic number: Smallest number of colors needed to color the vertices so that no two adjacent vertices have the same color (denoted $\chi(G)$)

The two invariants

- Chromatic number: Smallest number of colors needed to color the vertices so that no two adjacent vertices have the same color (denoted $\chi(G)$)
- 4CT, Brooks, Dirac, Gallai , $\cdots$

The two invariants

- Chromatic number: Smallest number of colors needed to color the vertices so that no two adjacent vertices have the same color (denoted $\chi(G)$)
- 4CT, Brooks, Dirac, Gallai , $\cdots$
- Possibly the most famous (and mysterious) graph invariant

The two invariants

- Chromatic number: Smallest number of colors needed to color the vertices so that no two adjacent vertices have the same color (denoted $\chi(G)$)
- 4CT, Brooks, Dirac, Gallai , $\cdots$
- Possibly the most famous (and mysterious) graph invariant
- What causes large chromatic number?

The two invariants

- Chromatic number: Smallest number of colors needed to color the vertices so that no two adjacent vertices have the same color (denoted $\chi(G)$)
- 4CT, Brooks, Dirac, Gallai , $\cdots$
- Possibly the most famous (and mysterious) graph invariant
- What causes large chromatic number?
- Hadwiger's conjecture 1943

The two invariants

- Chromatic number: Smallest number of colors needed to color the vertices so that no two adjacent vertices have the same color (denoted $\chi(G)$)
- 4CT, Brooks, Dirac, Gallai , $\cdots$
- Possibly the most famous (and mysterious) graph invariant
- What causes large chromatic number?
- Hadwiger's conjecture 1943
- Clique number: Largest size of a set of pairwise adjacent vertices (denoted $\omega(G)$)

The two invariants

- Chromatic number: Smallest number of colors needed to color the vertices so that no two adjacent vertices have the same color (denoted $\chi(G)$)
- 4CT, Brooks, Dirac, Gallai , $\cdots$
- Possibly the most famous (and mysterious) graph invariant
- What causes large chromatic number?
- Hadwiger's conjecture 1943
- Clique number: Largest size of a set of pairwise adjacent vertices (denoted $\omega(G)$)
- If $\omega = 1$, then $\chi = 1$. If $\omega = 2$, then $\chi =$???. (Mycielski, Zykov, Tutte , Kelly-Kelly, Erdős-Hajnal, Erdős)

A 124-year old riddle solved

Theorem (Appel, Haken – 1976)

Every planar graph has chromatic number at most 4.

Francis Guthrie

Perfect graphs

A graph G is perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G. (Berge 1961)

Perfect graphs

A graph G is perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G. (Berge 1961)

- Many fundamental graph classes are perfect. For example: bipartite graphs, comparability graphs, interval graphs, chordal graphs.

Perfect graphs

A graph G is perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G. (Berge 1961)

- Many fundamental graph classes are perfect. For example: bipartite graphs, comparability graphs, interval graphs, chordal graphs.
- Complement of a perfect graph is perfect (Lovasz 1972)

Perfect graphs

A graph G is perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G. (Berge 1961)

- Many fundamental graph classes are perfect. For example: bipartite graphs, comparability graphs, interval graphs, chordal graphs.
- Complement of a perfect graph is perfect (Lovasz 1972)
- Nice algorithmic properties (Grotschel, Lovasz, Schrijver 1981)

Perfect graphs

A graph G is perfect if $\chi(H) = \omega(H)$ for all induced subgraphs H of G. (Berge 1961)

- Many fundamental graph classes are perfect. For example: bipartite graphs, comparability graphs, interval graphs, chordal graphs.
- Complement of a perfect graph is perfect (Lovasz 1972)
- Nice algorithmic properties (Grotschel, Lovasz, Schrijver 1981)
- Forbidden induced subgraphs for the class of perfect graphs: Odd cycles of length at least 5 and their complements (Chudnovsky, Robertson, Seymour, Thomas 2006)

But what if graphs are not perfect? How about graphs that are not perfect but whose chromatic number is not very far off from their clique numbers?

χ-boundedness

A hereditary class $\mathcal{G}$ of graphs is said to be χ-bounded if there is a function f such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. In this case f is said to be a χ-bounding function for $\mathcal{G}$. (Gyàrfàs 1987)

χ-boundedness

A hereditary class $\mathcal{G}$ of graphs is said to be χ-bounded if there is a function f such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. In this case f is said to be a χ-bounding function for $\mathcal{G}$. (Gyàrfàs 1987)

Let $\mathcal{G}$ be a hereditary class of graphs.

χ-boundedness

A hereditary class $\mathcal{G}$ of graphs is said to be χ-bounded if there is a function f such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. In this case f is said to be a χ-bounding function for $\mathcal{G}$. (Gyàrfàs 1987)

Let $\mathcal{G}$ be a hereditary class of graphs.

- Is $\mathcal{G}$ χ-bounded?

χ-boundedness

A hereditary class $\mathcal{G}$ of graphs is said to be χ-bounded if there is a function f such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. In this case f is said to be a χ-bounding function for $\mathcal{G}$. (Gyàrfàs 1987)

Let $\mathcal{G}$ be a hereditary class of graphs.

- Is $\mathcal{G}$ χ-bounded?
- Is there a polynomial χ-bounding function for $\mathcal{G}$?

χ-boundedness

A hereditary class $\mathcal{G}$ of graphs is said to be χ-bounded if there is a function f such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. In this case f is said to be a χ-bounding function for $\mathcal{G}$. (Gyàrfàs 1987)

Let $\mathcal{G}$ be a hereditary class of graphs.

- Is $\mathcal{G}$ χ-bounded?
- Is there a polynomial χ-bounding function for $\mathcal{G}$?
- Is there a linear χ-bounding function for $\mathcal{G}$?

χ-boundedness

A hereditary class $\mathcal{G}$ of graphs is said to be χ-bounded if there is a function f such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. In this case f is said to be a χ-bounding function for $\mathcal{G}$. (Gyàrfàs 1987)

Let $\mathcal{G}$ be a hereditary class of graphs.

- Is $\mathcal{G}$ χ-bounded?
- Is there a polynomial χ-bounding function for $\mathcal{G}$?
- Is there a linear χ-bounding function for $\mathcal{G}$?
- What is the smallest χ-bounding function for $\mathcal{G}$?

Three theorems on χ-boundedness

Hole: Induced cycle of length at least 4

- The class of odd-hole-free graphs is χ-bounded. (Scott-Seymour 2016)

Three theorems on χ-boundedness

Hole: Induced cycle of length at least 4

- The class of odd-hole-free graphs is χ-bounded. (Scott-Seymour 2016)
- For every t, the class of graphs with no holes of length at least t is χ-bounded. (Chudnovsky-Scott-Seymour 2017)

Three theorems on χ-boundedness

Hole: Induced cycle of length at least 4

- The class of odd-hole-free graphs is χ-bounded. (Scott-Seymour 2016)
- For every t, the class of graphs with no holes of length at least t is χ-bounded. (Chudnovsky-Scott-Seymour 2017)
- For every t, the class of graphs with no odd holes of length at least t is χ-bounded. (Chudnovsky-Scott-Seymour-Spirkl 2019)

Three conjectures on χ-boundedness

1. Polynomial χ-bounding function is good for the Erdős-Hajnal conjecture.

Conjecture (Erdős-Hajnal 1989)

For every H, there exists $c > 0$ such that every n-vertex H-free graph has either a clique or stable set of size at least n^c.

Three conjectures on χ-boundedness

1. Polynomial χ-bounding function is good for the Erdős-Hajnal conjecture.

Conjecture (Erdős-Hajnal 1989)

For every H, there exists $c > 0$ such that every n-vertex H-free graph has either a clique or stable set of size at least n^c.

2. Esperet's conjecture: Every χ-bounded class is polynomially χ-bounded.

Three conjectures on χ-boundedness

1. Polynomial χ-bounding function is good for the Erdős-Hajnal conjecture.

Conjecture (Erdős-Hajnal 1989)

For every H, there exists $c > 0$ such that every n-vertex H-free graph has either a clique or stable set of size at least n^c.

2. Esperet's conjecture: Every χ-bounded class is polynomially χ-bounded.

3. Gyarfas-Sumner conjecture: For every tree T, the class of T-free graphs is χ-bounded.

(P_7, C_4, C_5)-free graphs

Theorem (Cameron-Huang-Penev-S.)

1. *Structure theorem for (P_7, C_4, C_5)-free graphs.*
2. *Optimization on (P_7, C_4, C_5)-free graphs.*
3. *If G is (P_7, C_4, C_5)-free, then $\chi(G) \leq 1.5\omega(G)$.*

Conjecture

If G is (P_5, C_5)-free, then $\chi(G) \leq \omega(G)^2$.

Polynomial χ-boundedness

Local complementation: Complement the neighborhood of a vertex
A graph H is a **vertex-minor** of G if H can be obtained from G by a sequence of vertex deletions and local complementations.

Theorem (Kim-Kwon-S.-Oum)

The class of graphs not containing C_n as a vertex-minor is polynomially χ-bounded.

Polynomial χ-boundedness

Local complementation: Complement the neighborhood of a vertex
A graph H is a **vertex-minor** of G if H can be obtained from G by a sequence of vertex deletions and local complementations.

Theorem (Kim-Kwon-S.-Oum)

The class of graphs not containing C_n as a vertex-minor is polynomially χ-bounded.

Motivated by:

Polynomial χ-boundedness

Local complementation: Complement the neighborhood of a vertex
A graph H is a **vertex-minor** of G if H can be obtained from G by a sequence of vertex deletions and local complementations.

Theorem (Kim-Kwon-S.-Oum)

The class of graphs not containing C_n as a vertex-minor is polynomially χ-bounded.

Motivated by:

Conjecture (Geelen 2009)

For any graph H, the class of graphs not containing H as a vertex-minor is χ-bounded.

A tool for χ-boundedness: Divisibility

Theorem (Chudnovsky-S.)

(P_5, C_5)-free graphs are 2-divisible.

A tool for χ-boundedness: Divisibility

Theorem (Chudnovsky-S.)

(P_5, C_5)-free graphs are 2-divisible.

Theorem (Chudnovsky-S.)

1. *$(bull, P_5$-free graphs are perfectly divisible.*
2. *$(bull, oddhole)$-free graphs are perfectly divisible.*

Conjecture (Hoàng-McDiarmid 2002)

Odd-hole-free graphs are 2-divisible.

Apexing

Let $\mathcal{G}$ be a graph class. Let $\mathcal{G}^{apex}$ denote the set of graphs G such that G contains a vertex v such that $G - v \in \mathcal{G}$.

Theorem (S.)

If every forbidden induced subgraph for $\mathcal{G}$ has at most c vertices, then every forbidden induced subgraph for $\mathcal{G}^{apex}$ has at most c^2 vertices.

Apexing

Let $\mathcal{G}$ be a graph class. Let $\mathcal{G}^{apex}$ denote the set of graphs G such that G contains a vertex v such that $G - v \in \mathcal{G}$.

Theorem (S.)

If every forbidden induced subgraph for $\mathcal{G}$ has at most c vertices, then every forbidden induced subgraph for $\mathcal{G}^{apex}$ has at most c^2 vertices.

Guoli Ding pointed out that the previous theorem holds for minors as well but it depends on the Graph Minors theorem.

Theorem

There exists a function f such that if every forbidden minor for $\mathcal{G}$ has at most c vertices, then every forbidden minor for $\mathcal{G}^{apex}$ has at most $f(c)$ vertices.

It would be interesting to explicitly come up with such a function.

Double-threshold graphs

Recently four Japanese authors studied a very natural generalization of threshold graphs, called double-threshold graphs. A graph is **double-threshold** if there exists $w : V(G) \to \mathbb{R}$ and $L, U \in \mathbb{R}$ such that uv is an edge if and only if $L \leq w(u) + w(v) \leq U$. The authors mentioned the problem of finding the forbidden induced subgraph characterization for the class but did not solve it.
My undergraduate student Deven Gill and I figured out some forbidden induced subgraphs for this class, and have a working conjecture

Conjecture

Every forbidden induced subgraph for the class of double-threshold graphs (other than cycles of length at least 5) has at most 7 vertices.

Questions and Problems

1. Is the class of (P_5, C_5)-free graphs χ-bounded by $f(x) = x^2$?

Questions and Problems

1. Is the class of (P_5, C_5)-free graphs χ-bounded by $f(x) = x^2$?
2. Scott-Seymour: $\chi \leq 2^{2^{\omega+2}}$ for odd-hole-free graphs. Better χ-bounding function?

Questions and Problems

1. Is the class of (P_5, C_5)-free graphs χ-bounded by $f(x) = x^2$?
2. Scott-Seymour: $\chi \leq 2^{2^{\omega+2}}$ for odd-hole-free graphs. Better χ-bounding function?
3. Hoang-McDiarmid conjecture: Odd-hole-free $=$ 2-divisible.

Questions and Problems

1. Is the class of (P_5, C_5)-free graphs χ-bounded by $f(x) = x^2$?
2. Scott-Seymour: $\chi \leq 2^{2^{\omega+2}}$ for odd-hole-free graphs. Better χ-bounding function?
3. Hoang-McDiarmid conjecture: Odd-hole-free $=$ 2-divisible.
4. Subquadratic χ-bounding function for graphs with cogirth at least 6?

Questions and Problems

1. Is the class of (P_5, C_5)-free graphs χ-bounded by $f(x) = x^2$?
2. Scott-Seymour: $\chi \leq 2^{2^{\omega+2}}$ for odd-hole-free graphs. Better χ-bounding function?
3. Hoang-McDiarmid conjecture: Odd-hole-free = 2-divisible.
4. Subquadratic χ-bounding function for graphs with cogirth at least 6?
5. Good χ-bounding function for graphs in which every hole and antihole has length at most 5?

THANK YOU