Toughness in pseudo-random graphs

Xiaofeng Gu
(University of West Georgia)

7th annual Mississippi Discrete Math Workshop

October 27, 2019

Background

- Random graph $G(n, p)$ model: a graph with n vertices is constructed by adding each edge randomly and independently with probability p.

Background

- Random graph $G(n, p)$ model: a graph with n vertices is constructed by adding each edge randomly and independently with probability p.
- A pseudo-random graph with n vertices of edge density p is a graph that behaves like a truly random graph $G(n, p)$.

Background

- Random graph $G(n, p)$ model: a graph with n vertices is constructed by adding each edge randomly and independently with probability p.
- A pseudo-random graph with n vertices of edge density p is a graph that behaves like a truly random graph $G(n, p)$.
- It was Thomason who first introduced the quantitative definition of pseudo-random graphs, by defining the term of jumbled graphs.

Background

- Random graph $G(n, p)$ model: a graph with n vertices is constructed by adding each edge randomly and independently with probability p.
- A pseudo-random graph with n vertices of edge density p is a graph that behaves like a truly random graph $G(n, p)$.
- It was Thomason who first introduced the quantitative definition of pseudo-random graphs, by defining the term of jumbled graphs.
- A graph G is (p, α)-jumbled (where $0 < p < 1 \leq \alpha$) if every vertex subset $U \subset V(G)$ satisfies:

$$\left| e(U) - p\binom{|U|}{2} \right| \leq \alpha |U|,$$

where p is the density and α controls the deviation.

Matrix and Eigenvalue

- Let G be a simple graph with vertices $v_1, v_2, \cdots, v_n$. The adjacency matrix of G, denoted by $A(G) = (a_{ij})$, is an $n \times n$ matrix such that $a_{ij} = 1$ if there is an edge between v_i and v_j, and $a_{ij} = 0$ otherwise.

Matrix and Eigenvalue

- Let G be a simple graph with vertices $v_1, v_2, \cdots, v_n$. The adjacency matrix of G, denoted by $A(G) = (a_{ij})$, is an $n \times n$ matrix such that $a_{ij} = 1$ if there is an edge between v_i and v_j, and $a_{ij} = 0$ otherwise.
- $\lambda_i(G)$ denotes the ith largest eigenvalue of $A(G)$. So we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Matrix and Eigenvalue

- Let G be a simple graph with vertices $v_1, v_2, \cdots, v_n$. The adjacency matrix of G, denoted by $A(G) = (a_{ij})$, is an $n \times n$ matrix such that $a_{ij} = 1$ if there is an edge between v_i and v_j, and $a_{ij} = 0$ otherwise.
- $\lambda_i(G)$ denotes the ith largest eigenvalue of $A(G)$. So we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.
- By Perron-Frobenius Theorem, λ_1 is always positive and $|\lambda_i| \leq \lambda_1$ for all $i \geq 2$. Let $\lambda = \max_{2 \leq i \leq n} |\lambda_i| = \max\{|\lambda_2|, |\lambda_n|\}$, that is, λ is the second largest absolute eigenvalue.

Pseudo-random graphs

Let G be a d-regular graph on n vertices.

- The expander mixing lemma: for every two subsets A and B of $V(G)$, $\left|e(A,B) - \frac{d}{n}|A||B|\right| \leq \lambda\sqrt{|A||B|}$, where $e(A,B)$ denotes the number of edges with one end in A and the other one in B (edges with both ends in $A \cap B$ are counted twice).

Pseudo-random graphs

Let G be a d-regular graph on n vertices.

- The expander mixing lemma: for every two subsets A and B of $V(G)$, $\left|e(A,B) - \frac{d}{n}|A||B|\right| \leq \lambda\sqrt{|A||B|}$, where $e(A,B)$ denotes the number of edges with one end in A and the other one in B (edges with both ends in $A \cap B$ are counted twice).
- By definition, G is $(d/n, \lambda)$-jumbled, and thus a kind of pseudo-random graph.

Pseudo-random graphs

Let G be a d-regular graph on n vertices.

- The expander mixing lemma: for every two subsets A and B of $V(G)$, $\left|e(A,B) - \frac{d}{n}|A||B|\right| \leq \lambda\sqrt{|A||B|}$, where $e(A,B)$ denotes the number of edges with one end in A and the other one in B (edges with both ends in $A \cap B$ are counted twice).
- By definition, G is $(d/n, \lambda)$-jumbled, and thus a kind of pseudo-random graph.
- A d-regular graph on n vertices with second largest absolute eigenvalue at most λ is called an (n, d, λ)-graph.

Research on pseudo-random graphs

- Extremal graph theory: An example
  Theorem (Krivelevich and Sudakov 2003)
  Let G be an (n, d, λ)-graph. If n is large enough and

$$\lambda < \frac{(\log \log n)^2}{1000 \log n (\log \log \log n)} d,$$

  then G is Hamiltonian.

Research on pseudo-random graphs

- Extremal graph theory: An example
  Theorem (Krivelevich and Sudakov 2003)
  Let G be an (n, d, λ)-graph. If n is large enough and

$$\lambda < \frac{(\log \log n)^2}{1000 \log n (\log \log \log n)} d,$$

  then G is Hamiltonian.
- Spectral graph theory: focus more on precise spectral bounds.

Toughness

- The toughness $t(G)$ of a connected graph G is defined as $t(G) = \min\{\frac{|S|}{c(G-S)}\}$, where the minimum is taken over all proper subset $S \subset V(G)$ such that $c(G-S) > 1$.

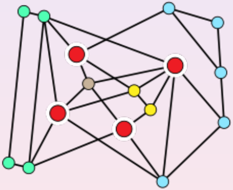

Figure: toughness $= 1$ (Picture from Wikipedia)

- G is t-tough if $t(G) \geq t$.

Toughness and cycles

- “Toughness at least 1” is a necessary condition for hamiltonicity.

Toughness and cycles

- "Toughness at least 1" is a necessary condition for hamiltonicity.
- Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness at least t_0 is Hamiltonian.

Toughness and cycles

- "Toughness at least 1" is a necessary condition for hamiltonicity.
- Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness at least t_0 is Hamiltonian.
- (False) Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness at least t_0 is pancyclic.

Toughness and cycles

- "Toughness at least 1" is a necessary condition for hamiltonicity.
- Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness at least t_0 is Hamiltonian.
- (False) Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness at least t_0 is pancyclic.
- Disproved by Bauer, van den Heuvel and Schmeichel (1995) who constructed t-tough triangle-free graphs for every t.

Toughness and cycles

- "Toughness at least 1" is a necessary condition for hamiltonicity.
- Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness at least t_0 is Hamiltonian.
- (False) Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness at least t_0 is pancyclic.
- Disproved by Bauer, van den Heuvel and Schmeichel (1995) who constructed t-tough triangle-free graphs for every t.
- Theorem (Alon 1995)
  For every t and g there exists a t-tough graph of girth greater than g.

Toughness in pseudo-random graphs

- Theorem (Alon 1995)
  For any connected d-regular graph G, $t(G) > \frac{1}{3}(\frac{d^2}{d\lambda+\lambda^2} - 1)$.

Toughness in pseudo-random graphs

- Theorem (Alon 1995)
  For any connected d-regular graph G, $t(G) > \frac{1}{3}(\frac{d^2}{d\lambda+\lambda^2} - 1)$.
- Together with the following result:
  Theorem(Lubotzky, Phillips and Sarnak 1988)
  There are infinitely many values of n with (n, d, λ)-graphs G_n on n vertices with $\lambda = \sqrt{d-1}$ such that the girth of G_n is at least $\frac{2}{3}\log_{d-1} n$.

Toughness in pseudo-random graphs

- Theorem (Alon 1995)
  For any connected d-regular graph G, $t(G) > \frac{1}{3}(\frac{d^2}{d\lambda+\lambda^2} - 1)$.
- Together with the following result:
  Theorem(Lubotzky, Phillips and Sarnak 1988)
  There are infinitely many values of n with (n, d, λ)-graphs G_n on n vertices with $\lambda = \sqrt{d-1}$ such that the girth of G_n is at least $\frac{2}{3}\log_{d-1} n$.
- Corollary (Alon 1995)
  There exists a positive constant C so that for every integer $g \geq 3$, there are infinitely many values of n with a graph G_n on n vertices whose girth is at least g so that $t(G_n) \geq n^{C/g}$.

Toughness in pseudo-random graphs

- Theorem (Alon 1995)
  For any connected d-regular graph G, $t(G) > \frac{1}{3}(\frac{d^2}{d\lambda+\lambda^2} - 1)$.
- Together with the following result:
  Theorem(Lubotzky, Phillips and Sarnak 1988)
  There are infinitely many values of n with (n, d, λ)-graphs G_n on n vertices with $\lambda = \sqrt{d-1}$ such that the girth of G_n is at least $\frac{2}{3}\log_{d-1} n$.
- Corollary (Alon 1995)
  There exists a positive constant C so that for every integer $g \geq 3$, there are infinitely many values of n with a graph G_n on n vertices whose girth is at least g so that $t(G_n) \geq n^{C/g}$.
- For every t and g there exists a t-tough graph of girth greater than g.

Improved results

- Theorem (Brouwer, 1995)
  For any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - 2$.

Improved results

- Theorem (Brouwer, 1995)
  For any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - 2$.
- There are infinitely many examples of graphs with $t \leq d/\lambda$.

Improved results

- Theorem (Brouwer, 1995)
  For any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - 2$.
- There are infinitely many examples of graphs with $t \leq d/\lambda$.
- In particular, Petersen graph P_{10}: $d = 3$ and $\lambda = 2$, while $t(P_{10}) = 4/3$.

Improved results

- Theorem (Brouwer, 1995)
  For any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - 2$.
- There are infinitely many examples of graphs with $t \le d/\lambda$.
- In particular, Petersen graph P_{10}: $d = 3$ and $\lambda = 2$, while $t(P_{10}) = 4/3$.
- Conjecture (Brouwer, 1995)
  For any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - 1$.

Improved results

- **Theorem** (Brouwer, 1995)
  For any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - 2$.
- There are infinitely many examples of graphs with $t \leq d/\lambda$.
- In particular, Petersen graph P_{10}: $d = 3$ and $\lambda = 2$, while $t(P_{10}) = 4/3$.
- **Conjecture** (Brouwer, 1995)
  For any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - 1$.
- **Theorem** (G. 2019+)
  For any connected d-regular graph G, $t(G) > \frac{d}{\lambda} - \sqrt{2}$.

Related conjecture

- Conjecture (Krivelevich and Sudakov 2003)
  There exists a positive constant C such that for large enough n, any (n, d, λ)-graph that satisfies $d/\lambda > C$ is Hamiltonian.

Related conjecture

- Conjecture (Krivelevich and Sudakov 2003)
  There exists a positive constant C such that for large enough n, any (n, d, λ)-graph that satisfies $d/\lambda > C$ is Hamiltonian.
- Recall: Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness greater than t_0 is Hamiltonian.

Related conjecture

- Conjecture (Krivelevich and Sudakov 2003)
  There exists a positive constant C such that for large enough n, any (n, d, λ)-graph that satisfies $d/\lambda > C$ is Hamiltonian.
- Recall: Conjecture (Chvátal, 1973)
  There exists some positive t_0 such that any graph with toughness greater than t_0 is Hamiltonian.
- Chvátal's conjecture implies the conjecture of Krivelevich and Sudakov.

Generalized connectivity

- Motivated by graph toughness, it is interesting to see how many vertices need to remove if we want a graph with a fixed number of components.

Generalized connectivity

- Motivated by graph toughness, it is interesting to see how many vertices need to remove if we want a graph with a fixed number of components.
- Given an integer $l \geq 2$, Chartrand, Kapoor, Lesniak and Lick defined the l-connectivity $\kappa_l(G)$ of a graph G to be the minimum number of vertices of G whose removal produces a disconnected graph with at least l components or a graph with fewer than l vertices.

Generalized connectivity

- Motivated by graph toughness, it is interesting to see how many vertices need to remove if we want a graph with a fixed number of components.
- Given an integer $l \geq 2$, Chartrand, Kapoor, Lesniak and Lick defined the l-connectivity $\kappa_l(G)$ of a graph G to be the minimum number of vertices of G whose removal produces a disconnected graph with at least l components or a graph with fewer than l vertices.
- When $l = 2$, it is the classical connectivity $\kappa(G)$.

Generalized connectivity

- Motivated by graph toughness, it is interesting to see how many vertices need to remove if we want a graph with a fixed number of components.
- Given an integer $l \geq 2$, Chartrand, Kapoor, Lesniak and Lick defined the l-connectivity $\kappa_l(G)$ of a graph G to be the minimum number of vertices of G whose removal produces a disconnected graph with at least l components or a graph with fewer than l vertices.
- When $l = 2$, it is the classical connectivity $\kappa(G)$.
- By definition, for a noncomplete connected graph G, we have $t(G) = \min_{2 \leq l \leq \alpha}\{\frac{\kappa_l(G)}{l}\}$ where α is the independence number of G.

Results

- Theorem (Fiedler 1973)
  For a d-regular graph, $\kappa \geq d - \lambda_2$.

Results

- Theorem (Fiedler 1973)
  For a d-regular graph, $\kappa \geq d - \lambda_2$.
- Theorem (Krivelevich and Sudakov 2006)
  For an (n, d, λ)-graph with $d \leq n/2$, $\kappa \geq d - \frac{36\lambda^2}{d}$.

Results

- **Theorem** (Fiedler 1973)
  For a d-regular graph, $\kappa \geq d - \lambda_2$.
- **Theorem** (Krivelevich and Sudakov 2006)
  For an (n, d, λ)-graph with $d \leq n/2$, $\kappa \geq d - \frac{36\lambda^2}{d}$.
- **Theorem** (G. 2019+)
  Let G be an (n, d, λ)-graph with $d \leq \alpha \cdot n$ for a constant $0 < \alpha < 1$. Let c be a constant with $c \geq \dfrac{1 + \sqrt{1 + \alpha + \frac{1}{\ell - 1}}}{1 - \alpha}$.
  Then the ℓ-connectivity of G satisfies

$$\kappa_\ell(G) \geq d - \frac{(c \cdot \lambda)^2}{d}.$$

Thank You